# CHIRAL SYMMETRY BREAKING:
# MESON AND NUCLEON MASSES

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1969

(Submitted November 7, 1968)

## ACKNOWLEDGMENTS

The author thanks B. Renner for practical encouragement and R. P. Feynman for his kind advisement.

# ABSTRACT

From consideration of the role in current algebra of the equal-time commutator involving an axial vector charge and divergence of the axial vector current, estimates of meson and nucleon scalar density matrix elements are given. The scalar densities, behaving as $(3, 3^*) + (3^*, 3)$ under the $SU(3) \times SU(3)$ algebra of vector and axial vector charges, participate in the $SU(3) \times SU(3)$ non-invariant part of an (abstracted) energy density. The meson scalar density matrix elements are estimated in connection with a $K\pi$, KK scattering length determination and a relation for the relevant symmetry breaking parameter $\varkappa$ which multiplies the octet algebraic scalar piece of the hadronic energy density. An application is given in which a $K\pi$ amplitude expansion, containing the information of a kaon scalar density term, is used in determining the low energy limits of a $K_{L_4}$ form factor. The estimates for the nucleon scalar density terms are somewhat inconclusive and emphasis is placed on discussion of various dispersive approaches involved.

# TABLE OF CONTENTS


| PART | TITLE | PAGE |
|---|---|---|
| I. | INTRODUCTION | 1 |
| II. | ALGEBRAIC PRELIMINARIES | 7 |
| III. | MESON SCALAR DENSITY TERMS, $K\pi$ AND KK SCATTERING LENGTHS AND THE SYMMETRY BREAKING PARAMETER $\varkappa$ | 10 |
| | 1) The $K\pi$ System | 11 |
| | 2) The KK System | 18 |
| IV. | DISCUSSION OF A $K_{L_4}$ FORM FACTOR | 23 |
| V. | NUCLEON SCALAR DENSITY TERMS | 30 |
| | 1) Low Energy Extrapolations | 32 |
| | 2) Discussion of Generalizations | 38 |
| VI. | CONCLUSION | 45 |
| APPENDICES | | |
| A. | AMPLITUDE SYMMETRY PROPERTIES IN MANDELSTAM INVARIANTS | 46 |
| B. | WEAK BORN TERM CONTRIBUTIONS | 49 |
| C. | FINITE ENERGY SUM RULE REPRESENTATION | 52 |
| D. | SOME DETAILS OF MASS DISPERSION REPRESENTATION | 55 |
| REFERENCES | | 59 |
| FIGURES | | 71 |

# I. INTRODUCTION

In this thesis an estimation is made of the nucleon and meson matrix elements of certain scalar densities $u_0$, $u_8$ which appear in the SU(3) $\times$ SU(3) non-invariant part of an "abstracted" energy density. Using the model algebra discussed in Section II, the energy density is given in a quark theory with an SU(3) $\times$ SU(3) invariant interaction and different iso-singlet and iso-doublet masses:

$$\theta_{00} = \theta^{x}_{00} + u_0 + \varkappa u_8 \, . \tag{1}$$

Here the symmetry breaking parameter $\varkappa$ multiplies the algebraic octet piece $u_8$, while the density $\theta^{x}_{00}$ denotes the part of $\theta_{00}$ which is invariant under the SU(3) $\times$ SU(3) algebra. The matrix elements of the scalar densities or "scalar density terms" are related to matrix elements of the equal-time commutator of an axial charge $A^i = \int dx^3 A^i_0$ and divergence of the axial vector current $D^i \equiv \partial^\mu A^i_\mu$,

$$[A^i, D^j] \tag{2}$$

where $i, j = 0, 1, \ldots 8$ are unitary spin indices. Matrix elements of (2) are then connected with the Fourier transforms of axial current and divergence time-ordered products which, in low energy limits and using specified extrapolations from vanishing four momenta, may be related to hadron scattering amplitudes for evaluation.

In Section III, estimation of s-wave $K\pi$ and KK scattering lengths will be made. The relation will be discussed of these threshold quantities to the meson scalar density terms as well as to an approximate determining relation for $\varkappa$:

$$\left(\frac{\sqrt{2} + \kappa}{\sqrt{2} - \frac{\kappa}{2}}\right) = \left(\frac{m_\pi^2}{m_K^2}\right)\left(\frac{f_\pi}{f_K}\right) \quad . \tag{3}$$

A typical evaluation to emerge from this section which shows the sizes of the meson scalar density terms is given by

$$\langle \pi^i | u_0 + \kappa u_8 | \pi^i \rangle = m_\pi^2 \ . \tag{4}$$

In Section IV, an expansion in Mandelstam invariants for the $\pi K$ amplitude (obtained from the previous section) will be used to discuss the rapid variation of a $K^+_{L_4}$ form factor due to a K-meson pole as well as the smooth variation of the $K_{L_3}$ - $K_{L_2}$ Callan-Treiman relation.

Finally in Section V, emphasis will be placed upon the various kinds of estimation procedures for the nucleon scalar density terms. Firm conclusions regarding the fraction of the nucleon mass represented by these terms await, at the least, better extrapolation techniques in the axial current divergence four momenta.

We now give a brief discussion of some previous attempts to relate chiral symmetry breaking with contributions to baryon and meson masses. Our motivation will be to gather expectations about matrix elements of the commutator (2) which contains the axial current divergence (or about the sizes of matrix elements of the scalar densities in (1) - a topic which will involve the model of the next section). For example, within Lagrangian models of various complexity (and contrivance) the behavior can be illustrated of particle masses under chiral symmetry limits where $D_i(x)$ vanishes.

Suppose one first deals with a single nucleon field $\psi(x)$. The axial current $A_\mu = \bar{\psi}\gamma_5\gamma_\mu\psi$ which is generated by a continuous "$\gamma_5$"

transformation $\psi \to e^{i\alpha\gamma_5}\psi$ is conserved for a massless nucleon as in

$$\theta_{00} = -\frac{i}{2}\bar{\psi}\underset{\sim}{\gamma}\cdot\overset{\leftrightarrow}{\underset{\sim}{\nabla}}\psi + g[(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5\psi)^2] . \tag{5}$$

At least in this chiral limit ($D(x) = 0$), one might expect the nucleon mass, abstracted from the model, to vanish. However, this possibility need not imply $\langle N|\theta_{00}|N\rangle = \langle N|\theta^x_{00}|N\rangle = 0$ because the states themselves can change from the physical case to the chiral invariant case (we have used the notation of eqn. (1) together with eqn. (5) and $|N\rangle$ denotes a physical nucleon state). On the other hand, the simple Nishijima[1] model illustrates the existence of a massless pion field which preserves chiral invariance of the energy density (or Lagrangian) in the presence of a finite nucleon mass:

$$\theta_{00} = -\frac{i}{2}\bar{\psi}\underset{\sim}{\gamma}\cdot\overset{\leftrightarrow}{\underset{\sim}{\nabla}}\psi + \frac{1}{2}[(\nabla_0\phi)^2 + (\underset{\sim}{\nabla}\phi)^2] + M\bar{\psi}e^{i\frac{g}{M}\gamma_5\phi}\psi . \tag{6}$$

(Note $M\bar{\psi}e^{i\frac{g}{M}\gamma_5\phi}\psi = M\bar{\psi}\psi + ig\bar{\psi}\gamma_5\psi + \ldots$, reproducing the mass and familiar coupling terms.) Under the transformations $\psi \to e^{i\alpha\gamma_5}\psi$ and $\phi \to \phi - (2M/g)\alpha$ $(\nabla_\mu\phi \to \nabla_\mu\phi)$, the conserved axial current is found to be $A_\mu = \bar{\psi}\gamma_5\gamma_\mu\psi - (2M/g)\nabla_\mu\phi$. Such a model could support the view that were the pion mass not zero, the divergence D would involve this quantity as a proportionality constant. (The "size" of the commutator $[A, D]$ might even be argued to be of the order of a pion mass squared.) In fact, in pseudo-vector theory, the $\sigma$-model[2] and phenomenological models[3] the model operator relation

$$D^i(x) = m_\pi^2 \frac{f_\pi}{\sqrt{2}} \phi_\pi^i(x) , \quad i = 1, 2, 3 \tag{7}$$

results, which exhibits $SU(2) \times SU(2)$ symmetry for a zero pion mass. Of course in (7), $D^i(x)$ may also vanish because $f_\pi \to 0$ in the chiral symmetry limit as a consequence of a vanishing nucleon mass from the Goldberger-Treiman relation $f_\pi \simeq \sqrt{2}\, MG_A/g_{\pi NN}$, where $G_A$ = the axial vector $\beta$-decay renormalization constant $\simeq 1.20$ and the pion-nucleon coupling constant is given by $g^2_{\pi NN} \simeq 4\pi\ (14.6)$. Finally, in a simple free quark (unitary triplet, spinor) model where, for example,

$$\theta_{00} = -\frac{i}{2}\bar{\psi}\underset{\sim}{\gamma}\cdot\overset{\leftrightarrow}{\underset{\sim}{\nabla}}\psi + u_0 \tag{8}$$

with $u^i = 2(3/2)^{1/2} m_q \bar{\psi}\frac{\lambda^i}{2}\psi$, $i = 0, \ldots 8$, the divergences $D_i = -2m_q(i\bar{\psi}\gamma_5\frac{\lambda^i}{2}\psi)$ vanish for a zero quark mass $m_q = 0$.

Let us now rephrase some of these observations in dispersion language. For illustration, take the neutron-proton axial vector matrix element $\langle p|A_\mu^{1+i2}|n\rangle = \bar{u}(p)\gamma_5[\gamma_\mu F_A(q^2) + q_\mu F_P(q^2)]u(n)$ which defines the divergence matrix element $\langle p|D^{1+i2}|n\rangle = -i\bar{u}(p)\gamma_5 u(n)\cdot D(q^2)$, where $D(q^2) = 2MF_A(q^2) + q^2F(q^2)$ with $q = n - p$. For a conserved axial current $D^{1+i2}(x) = 0$, so that $D(q^2) = 0$ and

$$F_P(q^2) = -\frac{2MF_A(q^2)}{q^2}\,,$$

$$\langle p|A_\mu|n\rangle = \bar{u}(p)\gamma_5[\gamma_\mu - \frac{2M}{q^2}q_\mu]F_A(q^2)u(n)\ . \tag{9}$$

At $q^2 \to 0$ a pole would appear in eqns. (9) with the quantum numbers of a massless pion unless the residue $2MF_A(0) = 2MG_A$ vanishes. One may then ask if in the product $MG_A$, the renormalization $G_A$

could be zero. From the Adler-Weisberger calculation[4] in the limit that $D^{1\pm i2}(x) = 0$, however, one finds $G_A^2 = 1$ not $G_A^2 = 0$. To see this result, write in the vanishing divergence limit the current algebraic identity,

$$q^\mu q^\nu T^{(-)}_{\mu\nu}(\nu, q^2) = \nu F_1^V(0) \tag{10}$$

where $T^{(-)}_{\mu\nu} = \frac{1}{2}(T^-_{\mu\nu} - T^+_{\mu\nu})$, $T^\pm_{\mu\nu} = -i \int d^4x\, e^{iq\cdot x} \langle p | T\{A_\mu^{1\mp i2}(x), A_\nu^{1\pm i2}(0)\} | p\rangle$, the invariant $\nu = p \cdot q$ and $F_1^V(0)$, the isovector proton electromagnetic form factor, satisfies $F_1^V(0) = 1$. In the RHS of (10), the commutator $\delta(x_0)[A_0^i(x), A_\mu^j(0)] = i\epsilon ijk\, \delta(x) V_\mu^k$ has been used; since the limit $q \to 0$ will eventually be taken, no additional gradient terms in this commutator will interfere with the argument. Following Fubini[5], we now write unsubtracted dispersion relations at fixed $q^2$ in the energy variable $\nu$ for the invariant amplitudes appearing in the decomposition (for spin-averaged proton), $T^{(-)}_{\mu\nu} = p_\mu p_\nu A + (p_\mu q_\nu + p_\nu q_\mu)B + q_\mu q_\nu C + g_{\mu\nu} D$; e.g., $A(\nu, q^2) = \frac{1}{\pi}\int \frac{d\nu'}{\nu' - \nu} a(\nu', q^2)$. Using the fact that the imaginary part of (10) is zero and that the crossing symmetry relations for imaginary parts are given by: $a(-\nu, q^2) = a(\nu, q^2)$, $b(-\nu, q^2) = -b(\nu, q^2)$, etc., one receives from (10),

$$-\frac{1}{\pi}\int a(\nu, q^2)d\nu = F_1^V(0). \tag{10'}$$

Let us now take $q^2 \to 0$ in (10)' and notice that the relation $\nu^2 a + 2\nu q^2 b + q^4 c + q^2 d = 0$ implies $\nu^2 a(\nu, 0) = 0$ and hence $a(\nu, 0) = 0$, except where there is the Born contribution $a^B(\nu, q^2) = -\frac{\pi}{2}\{\delta(\nu - \frac{q^2}{2}) + \delta(\nu + \frac{q^2}{2})\} F_A^2(q^2)$. Thus $G_A^2 = F_1^V(0) = 1$ emerges from (10)'.[6] So unless the nucleon mass M vanishes in the chiral limit, there will be a $1/q^2$ pole corresponding to a massless pion.

Given the energy density (1), a massless pion limit suggests that $\langle\pi|u_0 + \varkappa u_8|\pi\rangle$ (as well as $\langle\pi|\theta^x_{00}|\pi\rangle$) is proportional to $m_\pi^2$ in the physical case. In fact, a low energy limit directly applied to the scalar vertex as discussed in Section III leads to $\langle\pi|u_0 + \varkappa u_8|\pi\rangle = m_\pi^2$, where the state normalization $\langle p|p\rangle = (2\pi)^3\, 2p_0\, \delta(\underset{\sim}{0})$ is such that $\langle\pi|\theta_{00}|\pi\rangle_{\underset{\sim}{p}=0} = 2m_\pi^2$. Although there doesn't seem to be an analogous pole statement for the nucleon itself, it is of interest to estimate what fraction of the nucleon mass is represented by the quantity $\langle N|u_0 + \varkappa u_8|N\rangle$. In Section V, various current algebra dispersive techniques and extrapolation methods are discussed in this connection.

## II. ALGEBRAIC PRELIMINARIES

Matrix elements of the equal-time commutator $[A^i, D^j]$ may be related via a current algebraic identity (obtained by partial integration) to various current-target scattering amplitudes. In turn, these "weak" amplitudes may be related under certain assumptions to "strong" or hadronic scattering amplitudes for possible evaluation. We wish to identify this commutator with scalar density operators, certain members of which exist in a model expression for the hadronic energy density operator $\theta_{00}$. From particle matrix elements of these scalar densities, one then gets estimates for the SU(3) × SU(3) non-invariant pieces of particle masses.

To begin, algebraic properties of operators are abstracted from a quark prototype with different iso-doublet and iso-singlet masses and an SU(3) × SU(3) invariant interaction term. The energy density is given by

$$\theta_{00} = \theta_{00}^{x} + u_0 + \varkappa u_8 , \tag{1}$$

where in the model the scalar densities are given by $u^i = 2(3/2)^{1/2} m_0 \cdot \bar{\psi} \frac{\lambda^i}{2} \psi$; pseudo-scalar densities which will be connected to the axial divergences satisfy $v_i = 2(3/2)^{1/2} m_0 \cdot i \bar{\psi} \gamma_5 \frac{\lambda^i}{2} \psi$ and the dimensionless parameter $\varkappa = -\frac{\sqrt{2}}{3}(m_s - m_d)/m_0$ with $m_0 = \frac{1}{3} m_s + \frac{2}{3} m_d$. The vector and axial vector currents are given by $V_\mu^i = \bar{\psi} \gamma_\mu \frac{\lambda^i}{2} \psi$ and $A_\mu^i = \bar{\psi} \gamma_5 \gamma_\mu \frac{\lambda^i}{2} \psi$ and the charges $V_i = \int d^3x\, V_0^i$ and $A_i = \int d^3x\, A_0^i$ satisfy the equal-time SU(3) × SU(3) algebra:

$$
\begin{aligned}
[V_i, V_j] &= if_{ijk} V_k \\
[V_i, A_j] &= if_{ijk} A_k \\
[A_i, A_j] &= if_{ijk} V_k \ .
\end{aligned}
\tag{11}
$$

The scalar and pseudo-scalar densities $u_i$, $v_i$ transform as $(3^*, 3) + (3, 3^*)$ under the $SU(3) \times SU(3)$ algebra of vector and axial vector charges[7]:

$$
\begin{aligned}
[V_i, u_j] &= if_{ijk} u_k \\
[V_i, v_j] &= if_{ijk} v_k \\
[A_i, v_j] &= id_{ijk} u_k \\
[A_i, u_j] &= -id_{ijk} v_k \ .
\end{aligned}
\tag{12}
$$

All statements regarding the behavior under $SU(3) \times SU(3)$ of current densities can be checked in the model by using the canonical commutation relation $\{\psi^+_\alpha(\underset{\sim}{x}), \psi_\beta(\underset{\sim}{0})\} = \delta_{\alpha\beta}\delta(\underset{\sim}{x})$ so that $[\psi^+(\underset{\sim}{x})\Gamma\lambda\psi(\underset{\sim}{x}), \psi^+(\underset{\sim}{0})\Gamma'\lambda'\psi(\underset{\sim}{0})] = \delta(\underset{\sim}{x}) \cdot \psi^+[\Gamma\lambda, \Gamma'\lambda']\psi$, where $\Gamma\lambda$ denotes the direct product of dirac gamma and unitary spin matrices. The kinetic energy term in (1), $-\frac{i}{2}\bar{\psi}\underset{\sim}{\gamma}\cdot\overset{\leftrightarrow}{\underset{\sim}{\nabla}}\psi$, is $SU(3) \times SU(3)$ invariant and examples of $SU(3) \times SU(3)$ interactions include $A^{i=0}_\mu A^\mu_{i=0}$, $V^{i=0}_\mu V^\mu_{i=0}$, $u^i u_i + v^i v_i$ and $A^i_\mu A^\mu_i + V^i_\mu V^\mu_i$.

Using the expression (1) for $\theta_{00}$ and equations (12), $i[\int d^3x\,(\theta^x_{00} + u_0 + \varkappa u_8),\ A^i] = -i[A^i,\ \int d^3x\,(u_0 + \varkappa u_8)] = \int d^3x\, D^i$ so that we may identify,

$$D^i = -\,[d_{ii0} + \varkappa d_{ii8}]v^i,\quad i \neq 0, 8\,. \tag{13}$$

For example, $D^i = -\frac{1}{\sqrt{3}}(\sqrt{2} + \varkappa)v^i$ for $i = 1, 2, 3$ and $D^i = -\frac{1}{\sqrt{3}}(\sqrt{2} - \frac{\varkappa}{2})v^i$ for $i = 4, 5, 6, 7$. Now the commutator (2) can be written in terms of scalar densities; as an illustration we write out such an expression,

$$\begin{aligned}[A^{\pi^+},\ D^{\pi^-}] &= -\frac{1}{2\sqrt{3}}(\sqrt{2} + \varkappa)[A^{1+i2},\ v^{1-i2}] \\ &= -\frac{i}{3}(\sqrt{2} + \varkappa)(\sqrt{2}\,u_0 + u_8)\,.\end{aligned}$$

## III. MESON SCALAR DENSITY TERMS, $K\pi$ AND KK SCATTERING LENGTHS AND THE SYMMETRY BREAKING PARAMETER $\varkappa$

Using low-energy theorems from the $SU(2) \times SU(2)$ current algebra and Adler's PCAC self-consistency conditions,[8] Weinberg[9-a] has estimated the $\pi$-$\pi$ scattering lengths assuming that a linear expansion of the amplitude in Mandelstam invariants is approximately valid up to threshold. In order to determine some of the coefficients in the above expansion and without invoking the algebra of Section II, the isospin properties had to be specified of the scalar density term $\langle\pi|[A^i, D^j]|\pi\rangle$ (i, j are isospin indices here). It turns out that, within Weinberg's parameterization, specification alone of the pion vector charge matrix element and the requirement of no $I = 2$ contributions in $[A^i, D^j]$ completely determine[10] the value of this matrix element:

$$i\langle\pi^\ell|[A^i, D^j]|\pi^k\rangle = (m_\pi^2)\delta_{ij}\delta_{k\ell} \,. \tag{14}$$

On the other hand, one can arrive[9-b] at the same value by applying a low-energy limit directly to the scalar vertex defined by (14) upon using (the $SU(2) \times SU(2)$ version of) the algebra of Section II:

$$\begin{aligned}
i\langle\pi^\ell(p')|[A^i, D^j]|\pi^k(p)\rangle &= \frac{\delta_{ij}}{3}(\sqrt{2} + \varkappa)\,\langle\pi^\ell(p')|\sqrt{2}\,u_0 + u_8|\pi^k(p)\rangle \\
&= \frac{\delta_{ij}}{3}(\sqrt{2} + \varkappa)\cdot\frac{\sqrt{2}\,i(m_\pi^2 - p'^2)}{m_\pi^2 f_\pi}\int d^4x\, e^{ip'\cdot x}\langle 0|T\left\{\partial^\mu A_\mu^\ell(x)(\sqrt{2}\,u_0 + u_8)\right\}|\pi^k(p)\rangle \\
&\xrightarrow[(p'\to 0)]{} -\frac{\sqrt{2}\,i}{3 f_\pi}(\sqrt{2}+\varkappa)\langle 0|[A^\ell, \sqrt{2}\,u_0 + u_8]|\pi^k(p)\rangle\,\delta_{ij} \\
&= m_\pi^2\,\delta_{ij}\cdot\langle 0|D^\ell|\pi^k\rangle\left(m_\pi^2\frac{f_\pi}{\sqrt{2}}\right)^{-1} = (m_\pi^2)\,\delta_{ij}\,\delta_{k\ell} \,.
\end{aligned} \tag{15}$$

Here, the normalization condition $\langle \pi^i | D^i | 0 \rangle = m_i^2 \frac{f_i}{i\sqrt{2}}$ ($f_\pi = \sqrt{2}\, MG_A / g_{\pi NN}$) has been used in the reduction technique of the pion $\pi^\ell(p')$ for the $p' \to 0$ limit. In order for (14), an equation involving pion states of equal four-momenta, to follow from this exercise it must be assumed that the matrix element does not vary appreciably for zero-momenta of one state.

In this section, we generalize Weinberg's work and estimate s-wave $K\pi$ and KK scattering lengths as well as the concomitant meson scalar density matrix elements. In particular, the scalar density terms which appear in the KK case are estimated by established SU(3) × SU(3) commutators (11) and PCAC principles, just as in Weinberg's $\pi\pi$ estimation above. For the $K\pi$ case additional assumptions are needed and the scalar terms are determined by i) applying a low-energy limit to the scalar density vertex as in exercise (15), and ii) using an (approximate) determining relation for the relevant symmetry breaking parameter $\kappa$ which appeared in $\theta_{00}$. The scattering length determination for the KK case is consistent with this last procedure. Finally, the consistency of our use of the relation for $\kappa$ itself is discussed.

1) The $K\pi$ System

Here there are the two independent s-wave scattering lengths in the $I = 1/2$ and $I = 3/2$ states: $a^{1/2}$ and $a^{3/2}$ (the $\bar{K}\pi$ system is related to the $K\pi$ system by a charge conjugation operation or t-channel crossing symmetry). A linear expansion of the amplitude[11] $A^I(s, t, u; q^2, q^{2\prime}, p^2, p^{2\prime})$ in terms of invariants $s, t, u, q^2, q^{2\prime}, p^2$ and $p^{2\prime}$ is extrapolated to threshold assuming that there are no $J^P = 0^+$ bound states and that unitarity effects[12] do not lead to rapid variations of the amplitude at low energies. (Refer to Figure 1 for kinematical definitions).

We write the definite I-spin s-channel amplitudes in terms of amplitudes with definite t-channel charge-conjugation properties $\mathcal{C} = \pm$ satisfying crossing relations:

$$\begin{aligned} A^{(3/2)} &= A^{(+)} - A^{(-)} \\ A^{(1/2)} &= A^{(+)} + 2A^{(-)} \end{aligned} \tag{16}$$

where

$$\begin{aligned} A^{(\pm)}(s,t,u;q^2,q^{2\prime},p^2,p^{2\prime}) &= \pm A^{(\pm)}(u,t,s;q^{2\prime},q^2,p^2,p^{2\prime}) \\ &= \pm A^{(\pm)}(u,t,s;q^2,q^{2\prime},p^{2\prime},p^2). \end{aligned} \tag{17}$$

(For amplitude symmetry properties in the Mandelstam invariants, see Appendix A.) Using t-channel crossing symmetry (17) and the kinematical condition $s + t + u = q^2 + q^{2\prime} + p^2 + p^{2\prime}$, the linear expansions with constant coefficients can be written,

$$A^{(+)}(s,t,\ldots) = A + B(s+u) + C\,t + D(p^2 + p^{2\prime}), \tag{18}$$

$$A^{(-)}(s,t,\ldots) = A'(s-u)\,. \tag{19}$$

Incidentally, no terms of the form $B'(p^2 - p^{2\prime})$ or $C'(q^2 - q^{2\prime})$ appear in (19) because (17) applies to $p^2 \leftrightarrow p^{2\prime}$ and $q^2 \leftrightarrow q^{2\prime}$ separately.

In order to evaluate the coefficients A, B, C, D and A', we consider low-energy limits (vanishing four-momenta limits) in current-commutator identities of the form,

$$F^{ij}(s,t,\ldots) = q^{\mu\prime} q^{\nu} T^{ij}_{\mu\nu} - q^{\mu\prime} \int dx^4\, e^{iq'\cdot x} \langle p'|\, \delta(x_0)[A^i_\mu(x), A^j_0(0)]\,|\,p\rangle$$
$$+\, i \int dx^4\, e^{iq'\cdot x} \langle p'|\, \delta(x_0)[A^i_0(x), D^j(0)]\,|\,p\rangle \tag{20}$$

where

$$F^{ij} = -\, i \int dx^4\, e^{iq'\cdot x} \langle p'|\, T[D^i(x)\, D^j(0)]\,|\,p\rangle$$

and (21)

$$T^{ij}_{\mu\nu} = -i \int dx^4\, e^{iq'\cdot x} \langle p'|\, T[A^i_\mu(x)\, A^j_\nu(0)]\,|\,p\rangle$$

are Fourier transforms of the axial current and divergence time-ordered products. Observing the normalization condition $\langle \pi^i | D^i | 0 \rangle = m_i^2\ f_i/\sqrt{2}$, the following connection for low-energy applications will be assumed between the off-mass-shell boson-boson amplitude $A^{ij}$ and the weak amplitude $F^{ij}$,

$$A^{ij}(s,t,\ldots) = \frac{(q^{2\prime} - m_i^2)(q^2 - m_j^2)}{(m_i^2 \frac{f_i}{\sqrt{2}})\ (m_j^2 \frac{f_j}{\sqrt{2}})} \cdot F^{ij}(s,t,\ldots)\,. \tag{22}$$

We now take various low-energy limits in (20) for, say, $A^{(3/2)} = A(K^+\pi^+ \to K^+\pi^+)$ whose expansion is given by equations (16), (18) and (19).

(a) $q \to 0$ (or $q' \to 0$) with the other three particles on their mass shells (Adler's PCAC consistency condition[13]):

$$A^{(3/2)} \to A^{(3/2)}(m_K^2,\ m_\pi^2,\ m_K^2;\ 0,\ m_\pi^2,\ m_K^2,\ m_K^2) = 0$$

and,

$$A + 2m_K^2 (B + D) + m_\pi^2 C = 0 . \tag{23}$$

(b) $q'$, $q \to 0$ with the kaons on-mass-shell:

$$A^{(3/2)} \to A^{(3/2)}(m_K^2 + 2p\cdot q,\ 0,\ m_K^2 - 2p\cdot q;\ 0,\ 0,\ m_K^2,\ m_K^2)$$

$$= [A + 2m_K^2 (B + D)] - 4(p\cdot q)A' + 0(q^2,\ q'^2,\ q\cdot q')$$

$$= \frac{2i}{f_\pi^2}\langle K^+(p)|[A^{\pi^-}, D^{\pi^+}]|K^+(p)\rangle + \frac{2}{f_\pi^2}\langle K^+(p)|q^\mu V_\mu^3|K^+(p) + 0(q\cdot q')$$

and $$A' = -\frac{1}{2f_\pi^2} , \tag{24}$$

$$A + 2m_K^2 (B + D) = \frac{2}{3f_\pi^2}(\sqrt{2} + \varkappa)\langle K^+(p)|\sqrt{2}\, u_0 + u_8|K^+(p)\rangle . \tag{25}$$

Suppose we now use a low-energy limit directly applied to the vertex in order to evaluate the RHS of (25),

$$\langle K^+(p')|\sqrt{2}\, u_0 + u_8|K^+(p)\rangle = i\sqrt{2}\, \frac{(m_K^2 - p'^2)}{m_K^2 f_K} \cdot \int dx^4\, e^{ip'\cdot x}$$

$$\langle 0|T\left\{D^{K^-}(x)(\sqrt{2}\, u_0 + u_8)\right\}|K^+(p)\rangle$$

$$\xrightarrow[(p' \to 0)]{} - i\sqrt{2}\, f_K^{-1}\langle 0|[A^{K^-}, \sqrt{2}\, u_0 + u_8]|K^+(p)\rangle$$

$$= \frac{(3/2)m_K^2}{(\sqrt{2} - \frac{\varkappa}{2})} .$$

The RHS of (25) becomes,

$$\left(\frac{\sqrt{2}+\varkappa}{\sqrt{2}-\frac{\varkappa}{2}}\right)\ \frac{m_K^2}{f_\pi^2}\ . \tag{26}$$

However, this last expression can be greatly simplified by use of a determining relation for the parameter $\varkappa$.

$$\frac{\langle 0|D^{\pi^+}|\pi^-\rangle}{\langle 0|D^{K^+}|K^-\rangle} = \left(\frac{m_\pi^2}{m_K^2}\right)\left(\frac{f_\pi}{f_K}\right) = \left(\frac{\sqrt{2}+\varkappa}{\sqrt{2}-\frac{\varkappa}{2}}\right)\ \frac{\langle 0|v^{\pi^+}|\pi^-\rangle}{\langle 0|v^{K^+}|K^-\rangle}\ . \tag{27}$$

Of course in eqn. (27), we have merely used the expressions (13) which relate the axial current divergence to the pseudo-scalar densities $v^i$. If one now assumes the pseudo-scalar states to be nearly SU(3) symmetric[14], as motivated for example by success of the Gell-Mann-Okubo mass formula, the algebraic operators v connect the vacuum to states with unit norm so that

$$\left(\frac{\sqrt{2}+\varkappa}{\sqrt{2}-\frac{\varkappa}{2}}\right) = \left(\frac{m_\pi^2}{m_K^2}\right)\left(\frac{f_\pi}{f_K}\right)\ . \tag{3}$$

We note the relation between proximity of $\varkappa$ to the value $-\sqrt{2}$ and the smallness of the ratio $m_\pi^2/m_K^2$.[15] Inserting (26) with the relation (3) into (25),

$$A + 2m_K^2\,(B + D) = \frac{m_\pi^2}{f_\pi f_K}\ . \tag{25'}$$

Similarly, keeping the pions on-mass-shell and using various low-energy limits for the kaons:

(c) $p \to 0$ (or $p' \to 0$):

$$A^{(3/2)} \to A^{(3/2)}(m_\pi^2, m_K^2, m_\pi^2; m_\pi^2, m_\pi^2, m_K^2, 0) = 0$$

and

$$A + 2m_\pi^2 B + m_K^2 (C + D) = 0 \quad . \tag{28}$$

(d) $p, p' \to 0$:

$$A^{(3/2)} \to A^{(3/2)}(m_\pi^2 + 2p\cdot q, 0, m_\pi^2 - 2p\cdot q; m_\pi^2, m_\pi^2, 0, 0)$$

$$= [A + 2m_\pi^2 B] - 4(p\cdot q) A'$$

$$= \frac{2i}{f_K^2}\langle \pi^+ | [A^{K^-}, D^{K^+}] | \pi^+\rangle + \frac{1}{f_K^2}\langle \pi^+(q) | p^\mu (V_\mu^3 + \sqrt{3}\, V_\mu^8) | \pi^+(q)\rangle$$

$$= \frac{2}{3f_K^2}(\sqrt{2} - \frac{\varkappa}{2})\langle \pi^+(q) | \sqrt{2}\, u_0 - \frac{1}{2}u_8 + \frac{\sqrt{3}}{2}\, u_3 | \pi^+(q)\rangle + \frac{2(p\cdot q)}{f_K^2}$$

and

$$A' = -\frac{1}{2f_K^2} \quad , \tag{29}$$

$$A + 2m_\pi^2 B = \frac{m_K^2}{f_\pi f_K} \quad . \tag{30}$$

Again, in eqn. (30), a low-energy limit together with the determining relation for $\varkappa$ has been used to evaluate the scalar density term.

From (24) and (29), simultaneous pion and kaon low-energy limits would require the (approximately valid) equality[16], $f_\pi = f_K \equiv f$. Using the independent equations (23), (24), (25)', (28) and (30) to determine[17] the five constants A', A, B, C and D, we find

$$A' = -\frac{1}{2f^2}, \quad A = (m_K^2 + m_\pi^2)\frac{1}{f^2}, \quad B = -\frac{1}{2f^2}, \quad C = -\frac{1}{f^2}, \quad D = 0. \tag{31}$$

The s-wave scattering lengths are given[18)] in terms of the threshold amplitude by,

$$-8\pi(m_K + m_\pi)a^I = A^I(\,(m_K + m_\pi)^2, 0, (m_K - m_\pi)^2;\, m_\pi^2, m_\pi^2, m_K^2, m_K^2)\,.$$

At threshold, $A_{th}^{(+)} = A + 2(m_K^2 + m_\pi^2)B + 2m_K^2 D = 0$ and

$A_{th}^{(-)} = 4m_K m_\pi A' = -2m_K m_\pi \frac{1}{f^2}$ , so that using (16),

$$\begin{aligned} a^{1/2} &= 2\left(\frac{m_K}{m_K + m_\pi}\right) L \simeq 2L \simeq 0.22\ m_\pi^{-1}\ , \\ a^{3/2} &= -\left(\frac{m_K}{m_K + m_\pi}\right) L \simeq -L \simeq -0.11\ m_\pi^{-1}\ , \end{aligned} \tag{32}$$

where $\quad L = \frac{m_\pi}{4\pi f^2} \simeq 0.11\ m_\pi^{-1}$ .

These results for $a^{1/2,\,3/2}$ are identical to the ones obtained from Weinberg's "heavy target" formula[9)] for target-pion scattering in which $A_{th}^{(+)}$ is omitted:

$$a_W^I = -\left(\frac{m_K}{m_K + m_\pi}\right) [I(I+1) - 3/4 - 2]\, L\ .$$

(In this case, the target particle is the kaon.) In fact, if only limits (a) and (b) are used, $A_{th}^{(+)}/A_{th}^{(-)} = 0(m_\pi/m_K)$. The use of all the equations ($f_\pi = f_K$), however, gives $A_{th}^{(+)} = 0$ .

2) The KK System

There is only one s-wave scattering length $a^{(I=1)}$ since $a^{(0)}$ vanishes by Bose statistics. Applying to the I = 0, 1 s-channel amplitudes: (i) Bose statistics in the s-channel, and (ii) PT invariance which says invariance under $(s,t,u; q^2, q^{2\prime}, p^2, p^{2\prime}) \leftrightarrow (s,t,u; q^{2\prime}, q^2, p^{2\prime}, p^2)$, or (ii') t-channel crossing symmetry, one finds

$$A^{(1)} = a + b\,(u+t) + cs \quad \text{and} \quad A^{(0)} = a'\,(u-t)\,. \tag{33}$$

Using low-energy limits in commutator identities to determine the coefficients:

(a) $q \to 0$ (consistency condition):

$$A^{(1)} \to A^{(1)}(m_K^2, m_K^2, m_K^2; 0, m_K^2, m_K^2, m_K^2) = 0$$

and
$$a + 2m_K^2\,b + m_K^2\,c = 0\;. \tag{34}$$

(b) $q', q \to 0$ (or $p', p \to 0$):

$$A^{(1)} \to A^{(1)}(m_K^2 + 2p\cdot q,\ 0,\ m_K^2 - 2p\cdot q; 0, 0, m_K^2, m_K^2)$$

$$= [a + m_K^2\,(b+c)] + 2(p\cdot q)(c-b)$$

$$= \frac{2i}{f_K^2}\langle K^+(p)|\,[A^{K^-}, D^{K^+}]\,|K^+(p)\rangle + \frac{4(p\cdot q)}{f_K^2}\;,$$

and
$$c - b = 2/f_K^2\;, \tag{35}$$

$$a + m_K^2(b+c) = \frac{2}{3f_K^2}(\sqrt{2} - \frac{\varkappa}{2})\langle K^+(p)|\sqrt{2}\, u_0 - \frac{1}{2}\, u_8 + \frac{\sqrt{3}}{2} u_3 | K^+(p)\rangle \ . \qquad (36)$$

And

$$A^{(0)} \to m_K^2\, a' - 2(p\cdot q)\, a'$$

$$= \frac{2i}{f_K^2}\left\{2\langle K_0|[A^{K^-}, D^{K^+}]|K_0\rangle - \langle K^+|[A^{K^-}, D^{K^+}]|K^+\rangle\right\}$$

$$+ \ (0)\ (p\cdot q)$$

or

$$a' = 0 \ , \qquad (37)$$

$$a' = \frac{2}{3f_K^2}(\sqrt{2} - \frac{\varkappa}{2})\langle K^+(p)|\sqrt{2}\, u_0 - \frac{1}{2}\, u_8 - \frac{3\sqrt{3}}{2} u_3 | K^+(p)\rangle \ . \qquad (38)$$

(c) $p'$, $q' \to 0$ (or $p$, $q \to 0$):

$$A^{(1)} \to A^{(1)}(0, m_K^2 + 2p\cdot q', m_K^2 - 2p\cdot q'; m_K^2, 0, m_K^2, 0) = (a + 2m_K^2\, b)$$

$$= \frac{2i}{f_K^2}\langle 0|[A^{K^-}, D^{K^-}]|K^+(p)\, K^+(-p)\rangle = 0 \ .$$

So if there is no $Y = 2$ component in the commutator $[A^i, D^j]$,[7)]

$$a + 2m_K^2\, b = 0 \ . \qquad (39)$$

We note that Eqs. (35), the kaon vector charge matrix element, and (39), the requirement of no $Y = 2$ scalar density, determine the scalar density matrix element in (36): the right-hand side of (36) equals

$2m_K^2/f_K^2$. On the other hand, using the low-energy limit to evaluate the matrix element in (36) by taking one kaon four momentum to zero yields this same answer as well as confirming the equivalence of (37) and (38). Of course, this observation is entirely similar to one made previously for the $\pi\pi$ system.

Equations (34), (35), (37), and (39) give

$$a' = 0, \quad a = \frac{4m_K^2}{f_K^2}, \quad b = -\frac{2}{f_K^2}, \quad \text{and} \quad c = 0 \; . \tag{40}$$

So $A^{(1)}(s,t,u,\dots) = \frac{2}{f_K^2}[2m_K^2 - (u+t)]$ (while $A^{(0)}(s,t,\dots) = 0$ + quadratic terms in invariants) and using

$$-32\pi\, m_K\, a^{(1)} = A_{th}^{(1)} = A^{(1)}(4m_K^2, 0, 0; m_K^2, m_K^2, m_K^2, m_K^2) \, ,$$

$$a^{(1)} = -\frac{m_K}{8\pi f_K^2} \approx -\frac{1}{6}\, m_\pi^{-1} \; . \tag{41}$$

A linear expansion (33) for the KK amplitudes $A^{0,1}$ is plausible because for this $Y = 2$ system there are no unphysical threshold effects. However, such an expansion for the $Y = 0$ $K\bar{K}$ system (which can be related by crossing to KK) would not be expected to be valid since there are considerable unphysical thresholds for both $I = 0$ and $I = 1$ lying below the elastic one at $s_{th} = 4m_K^2 = 0.98\ (\text{BeV})^2$. Small $J^{PG} = 0^{++}, 0^{+-}$ effects may permit a calculation of s-wave $K\bar{K}$ scattering lengths. But the experimental situation reveals significant $I = 0, 1$ $K\bar{K}$ effects although there exist a variety of interpretations in the fits:[19)] non-resonance -- positive real scattering length, bound system - complex scattering length or resonance just above threshold.

Let us note that the ratio of the scalar density term coming from (14), $\frac{1}{3}(\sqrt{2}+\varkappa)\langle\pi|\sqrt{2}\,u_0 + u_8|\pi\rangle = m_\pi^2$, and from (36), $\frac{1}{3}(\sqrt{2}-\frac{\varkappa}{2})\langle K^+|\sqrt{2}\,u_0 - \frac{1}{2}u_8 + \frac{\sqrt{3}}{2}u_3|K^+\rangle = m_K^2$, together with SU(3) symmetry for pseudoscalar states can be evaluated without the necessity of a low-energy limit as:

$$\frac{(\sqrt{2}+\varkappa)}{(\sqrt{2}-\frac{\varkappa}{2})}\cdot\frac{\langle\pi|\sqrt{2}\,u_0+u_8|\pi\rangle}{\langle K^+|\sqrt{2}\,u_0-\frac{1}{2}u_8+\frac{\sqrt{3}}{2}u_3|K^+\rangle}$$

$$=\left(\frac{\sqrt{2}+\varkappa}{\sqrt{2}-\frac{\varkappa}{2}}\right)\left\{\frac{\alpha+(\frac{1}{\sqrt{3}})\beta}{\alpha-\frac{1}{2}(-\frac{1}{2\sqrt{3}})\beta+\frac{\sqrt{3}}{2}(\frac{1}{2})\beta}\right\}$$

$$=\left(\frac{\sqrt{2}+\varkappa}{\sqrt{2}-\frac{\varkappa}{2}}\right)=\frac{m_\pi^2}{m_K^2}\quad. \tag{42}$$

In this expression $\alpha, \beta$ have resulted from the independent reduced matrix elements of the scalar densities and the numerical factors are merely SU(3) symmetry $d_{ijk}$ coefficients. Equation (42) serves as a consistency check ($f_\pi = f_K$) on our previously derived eqn. (3) which involved a vacuum to single-particle state transition matrix element. Recently, Glashow and Weinberg (Phys. Rev. Letters 20, 224 (1968)) also wrote down an equation which resembles equation (27) (written down by the author in November, 1966). In fact, their procedure which involves pseudo-scalar dominance of spectral functions can be shown to be equivalent to PCAC applied at various pseudo-scalar vertices. This chapter, of course, has been concerned with using and demonstrating the consistency of the approximate eqn. (3) within the framework of a scattering length determination. The agreement of Weinberg's heavy target (kaon) formula with our $\pi K$ scattering lengths is just reflected in the smallness of the pion-kaon mass (squared) ratio of equation (3).

Finally as the result of our discussion of the interplay of scattering length and scalar density term estimates, we exhibit the low-energy limit directly applied to the arbitrary pseudo-scalar meson matrix element.

$$\langle M^i | u_0 + \kappa u_8 | M^i \rangle = -\frac{i\sqrt{2}}{f_i} \langle 0 | [A^i, \theta_{00}] | M_i \rangle$$

$$= m_i^2 \cdot \langle 0 | D^i | M^i \rangle \left(m_i^2 \frac{f_i}{\sqrt{2}}\right)^{-1} = m_i^2 \,. \tag{43}$$

## IV. DISCUSSION OF A $K_{L_4}$ FORM FACTOR

Using the expansion for the $K\pi$ scattering amplitude $A_{K^+\pi^+} = A^{(3/2)}$ obtained in the last section, it is shown that the soft pion theorems on $K_{L_4}$ decays are consistent with the Callan-Treiman relation on $K_{L_3}$ decays. We will exhibit the explicit relation for a $K_{L_4}$ form factor which, due to a K-meson pole, varies rapidly for simultaneous low-energy limits in the two pion four-momenta - the scalar density term which is contained in the expression for $A_{K\pi}$ is the pole residue. We now give some background related to this discussion.

Using a low-energy limit for the pion $\pi^o(p)$ in the $K^+_{L_3}$ matrix element

$$\langle \pi^o(p) | V_\mu^{K^-} | K^+(k) \rangle = -\frac{1}{2} [f_+(k+p)_\mu + f_-(k-p)_\mu] \tag{44}$$

$$f_\pm = f_\pm(k^2,\ p^2,\ \Delta^2 = (k-p)^2), \quad V_\mu^{K^-} = V_\mu^{\frac{4-i5}{\sqrt{2}}}$$

which is then connected with the $K^+_{L_2}$ matrix element

$$\langle 0 | A_\mu^{K^-} | K^+ \rangle = i \frac{f_K}{\sqrt{2}} k_\mu \ , \tag{45}$$

one receives the Callan-Treiman relation[20-a],

$$\frac{f_K}{f_\pi} = \left\{ f_+(m_K^2,\ p^2 = 0,\ \Delta^2 = m_K^2) + f_-(m_K^2, p^2 = 0, \Delta^2 = m_K^2) \right\}. \tag{46}$$

Carrying out the procedure described,

$$\langle\pi^0(p)|\,V_\mu^{K^-}\,|K^+(k)\rangle = \frac{i(m_\pi^2 - p^2)}{m_\pi^2\,\frac{f_\pi}{\sqrt{2}}}\Big\{-\int dx^4\, e^{ip\cdot x}\delta(x_0)\langle 0|\,[A_0^{\pi^0}(x), V_\mu^{K^-}]$$

$$|K^+\rangle - i p^\nu \int dx^4\, e^{ip\cdot x}\theta(x_0)\langle 0|\,[A_\nu^{\pi^0}(x),\; V_\mu^{K^-}]\,|K^+\rangle\Big\}$$

$$\xrightarrow[(p\to 0)]{} -\frac{\sqrt{2}\,i}{f_\pi}\langle 0|\,[A^{\pi^0}, V_\mu^{K^-}]\,|K^+\rangle = \frac{i}{\sqrt{2}\,f_\pi}\langle 0|\,A_\mu^{K^-}\,|K^+\rangle\,.$$

Equating coefficients of $k_\mu$, equation (46) follows. In the same manner[20-a], low-energy limits connect the $K^+_{L_4}$ matrix element (the process $K^+ \to \pi^+\pi^-\ell^+\nu$)

$$\langle\pi^+(p)\pi^-(q)|\,A_\mu^{K^-}\,|K^+(k)\rangle = \frac{i}{\sqrt{2}\,m_K}[F_1(p+q)_\mu + F_2(p-q)_\mu + F_3(k-p-q)_\mu]$$

$$F_{1,2,3} = F_{1,2,3}(k^2, p^2, q^2, k\cdot p, k\cdot q, p\cdot q) \tag{47}$$

with the $K^+_{L_3}$ matrix element (44):

(a) $p \to 0$, $q^2 = m_\pi^2$, $k^2 = m_K^2$ ;

$$\langle\pi^+(0)\pi^-(q)|\,A_\mu^{K^-}\,|K^+(k)\rangle = -\frac{i\sqrt{2}}{f_\pi}\langle\pi^-|\,[A^{\pi^-}, A_\mu^{K^-}]\,|K^+\rangle = 0$$

coefficient of $k_\mu$: $\qquad F_3(m_K^2,\ 0,\ m_\pi^2,\ 0,\ k\cdot q,\ 0) = 0 \qquad (48)$

coefficient of $q_\mu$: $F_1(m_K^2, 0, m_\pi^2, 0, k\cdot q, 0) - F_2(m_K^2, 0, m_\pi^2, 0, k\cdot q, 0)$

$$= F_3(m_K^2, 0, m_\pi^2, 0, k\cdot q, 0)\ . \qquad (49)$$

and

(b) $q \to 0,\ p^2 = m_\pi^2,\ k^2 = m_K^2$ ;

$$\langle \pi^+(p)\pi^-(0)| A_\mu^{K^-} | K^+(k)\rangle = \frac{i\sqrt{2}}{f_\pi} \langle \pi^0(p)| V_\mu^{K^-} | K^+(k)\rangle$$

coefficient of $k_\mu$: $F_3(m_K^2, m_\pi^2, 0, k\cdot p, 0, 0)$

$$= -m_K f_\pi^{-1} \left\{ f_+(m_K^2, m_\pi^2, \Delta^2) + f_-(m_K^2, m_\pi^2, \Delta^2) \right\} \qquad (50)$$

coefficient of $p_\mu$: $F_1(m_K^2, m_\pi^2, 0, k\cdot p, 0, 0)$

$$+ F_2(m_K^2, m_\pi^2, 0, k\cdot p, 0, 0) - F_3(m_K^2, m_\pi^2, 0, k\cdot p, 0, 0)$$

$$= -m_K f_\pi^{-1} \left\{ f_+(m_K^2, m_\pi^2, \Delta^2) - f_-(m_K^2, m_\pi^2, \Delta^2) \right\}\ . \qquad (51)$$

In extrapolation from vanishing pion four-momenta, one may invoke "smoothness" assumptions and neglect possible variations in the form factors $F_1$ and $F_2$. The same kind of smoothness assumption allows extrapolation of the Callan-Treiman relation (46) from p = 0 to the physical region $p^2 = m_\pi^2$, $0 \le \Delta^2 \le (m_K - m_\pi)^2$. Then from (a) and (b) above, $F_1$ and $F_2$ are replaced by constants which satisfy

$F_1 = F_2 \simeq -m_K f_\pi^{-1} f^+(\Delta^2 = 0)$, leading to good agreement with the measured rate of $K^+_{L_4}$ decay.[21] On the other hand, by comparing eqns. (48) and (50) for the form factor $F_3$ there must be considerable variation for the two limits $p \to 0$ and $q \to 0$. Weinberg[21] pointed out that such variation in $F_3$ was accounted for by a nearby singularity: the K-meson pole of Figure 2. However the precise incorporation of this observation has led to some confusion. Berman and Roy[22] have pointed out that Weinberg's neglect of a certain scalar density term ("$\sigma$-commutator") invalidates his expression for $F_3$ in the limit for which p and q are both small. But without an explicit expression for the $\pi K$ amplitude in the K-meson pole term of $F_3$, they lose certain terms in their argument and predict that the RHS of (50), like (48), should be zero so that $\xi = f_-(m_K^2, m_\pi^2, \Delta^2)/f_+(m_K^2, m_\pi^2, \Delta^2) \approx -1$.

Using our linear expansion in Mandelstam invariants for the off-mass-shell $\pi K$ amplitude, the residue of the K-meson pole term, we arrive at the conclusion

$$1 \simeq \frac{f_K}{f_\pi} \simeq \left\{ f_+(m_K^2,\ m_\pi^2,\ \Delta^2) + f_-(m_K^2,\ m_\pi^2,\ \Delta^2) \right\} . \tag{52}$$

When this equation is compared with the Callan-Treiman relation (46), it implies the consistency of neglecting variations for $f_\pm$ due to the change $p^2 = 0$ to $p^2 = m_\pi^2$ in the pion mass squared variable.[23]

We begin by writing $F_3$ as the sum[22), 24] of a pole term and a constant,

$$F_3(m_K^2, p^2, q^2, k\cdot p, k\cdot q, p\cdot q) = -m_K f_K \cdot \frac{A_{K^+\pi^+}(k-p-q, p; k, -q)}{[m_K^2 - (k-p-q)^2]} + C. \tag{53}$$

That is, smoothness is assumed for $F_3$ except for the K-meson pole contribution. We then insert, using equations (31), the expansion for $A_{K^+\pi^+} = A^{(3/2)}$,

$$A_{K^+\pi^+}(k - p - q,\ p;\ k,\ -q) = f^{-2}\left\{m_K^2 + m_\pi^2 - (u + t)\right\}$$

$$= f^{-2}\left\{m_K^2 + m_\pi^2 - (k^2 + q^2 + 2p^2 + 2p\cdot q - 2k\cdot p)\right\} \quad (54)$$

(here the decay constants satisfy $f_K \simeq f_\pi \equiv f$) so that,

$$F_3(m_K^2, p^2, q^2, k\cdot p, k\cdot q, p\cdot q) = -m_K f^{-1} \cdot \frac{[m_\pi^2 - q^2 - 2p^2 - 2p\cdot q + 2k\cdot p]}{[m_K^2 - (k - p - q)^2]} + C. \quad (55)$$

Recall that the expansion (54) has been constructed to incorporate all of Adler's consistency conditions (for example, for $q \to 0$ or for $p \to 0$ with all other particles on-mass-shell) as well as an evaluation in the low-energy limit of the relevant charge-divergence commutator (25) between kaon states.

We now use eqn. (55) for $F_3$ together with constant $F_1$, $F_2$ in the indicated low-energy limits which relate as before the $K^+_{L_4}$ and $K^+_{L_3}$ vertices:

(a) $p \to 0$, $q^2 = m_\pi^2$, $k^2 = m_K^2$ .

$$\text{coefficient of } k_\mu: \qquad C = 0\ , \quad (56)$$

$$\text{coefficient of } q_\mu: \qquad F_1 - F_2 = C \quad (57)$$

and

(b) $q \to 0,\ p^2 = m_\pi^2,\ k^2 = m_K^2$ .

coefficient of $k_\mu$: $(m_K f^{-1} - C)$

$$= m_K f^{-1}\left\{ f_+(m_K^2, m_\pi^2, \Delta^2) + f_-(m_K^2, m_\pi^2, \Delta^2)\right\} \tag{58}$$

coefficient of $p_\mu$: $F_1 + F_2 + (m_K f^{-1} - C)$

$$= -m_K f^{-1}\left\{ f_+(m_K^2, m_\pi^2, \Delta^2) - f_-(m_K^2, m_\pi^2, \Delta^2)\right\} . \tag{59}$$

Using (56), our result eqn. (52) follows from (58) where now[25)]

$$F_3(m_K^2,\ m_\pi^2,\ 0,\ k\cdot p,\ 0,\ 0) = -m_K f^{-1} . \tag{60}$$

In other words, using expansion (54) in the residue of the K-meson pole (53), one finds the residue to vanish at $q \to 0,\ p^2 = m_\pi^2,\ k^2 = m_K^2$ as required by Adler's consistency condition but in a definite way, namely like $\sim f^{-2}[m_K^2 - (k - p - q)^2]$, so that the entire pole term $F_3$ is non-zero.

Finally, we write down the equation for $F_3$ in the <u>simultaneous</u> limits $p, q \sim 0$. Using (55) when both p and q are small (and quadratic terms in these variables may be neglected) we get the <u>explicit</u> expression:

$$-F_3(m_K^2, p^2, q^2, k\cdot p, k\cdot q, p\cdot q) \underset{p,q\approx 0}{\approx} \frac{m_K}{2f}\left\{1 - \frac{k\cdot(q-p)}{k\cdot(q+p)}\right\} + m_K f \cdot \frac{A_{K^+\pi^+}(q,\ p=0)}{[m_K^2 - (k-p-q)^2]} . \tag{61}$$

Directly from (54), $A_{K^+\pi^+}(p, q{=}0) = \frac{2}{3f_\pi^2}(\sqrt{2}+\varkappa)\langle K^+|\sqrt{2}\, u_0 + u_8|K^+\rangle$ $= m_\pi^2 f^{-2}$, where the symmetry-breaking parameter $\varkappa$ satisfies (3), $(\sqrt{2}+\varkappa)/(\sqrt{2}-\frac{\varkappa}{2}) \simeq (m_\pi^2/m_K^2)(f_\pi/f_K)$. The first piece in the RHS of (61) is the quantity retained by Weinberg[21)], but there is an additional (infinite) pole term whose residue he neglected. Now the first quantity, by itself, gives for $F_3$ the correct separate limits $p \to 0$, $q^2 = m_\pi^2$, $k^2 = m_K^2$ and $q \to 0$, $p^2 = m_\pi^2$, $k^2 = m_K^2$. This is merely due to the fact that in these limits the consistency condition eliminates the entire second quantity in the RHS of (61).

Finally we note that the expression (53) for $F_3$ $(C = 0)$ can directly serve as a parameterization in $K_{\mu_4}$ decay (whereas in $K_{e_4}$ decay this coefficient multiplied the negligible lepton momentum).

## V. NUCLEON SCALAR DENSITY TERMS

We begin by writing a current identity like (20) which provides the basis for a relation between weak amplitudes and the equal-time commutator $[A^i, D^j]$. For this nucleon scalar density discussion it is convenient to consider at the outset a version of the identity with symmetric unitary spin indices and with the kinematics displayed in Figure 3:

$$\int dx^4 e^{iK\cdot x} \langle p' | i\delta(x_0)[A_0^i(\tfrac{x}{2}), D^j(-\tfrac{x}{2})] | p\rangle$$
$$= F^{(+)}(\nu, t, w, v) - q'_\mu q_\nu T^{(+)}_{\mu\nu}(\nu, t, w, v) . \tag{62}$$

This identity with proton states involves the combinations $F^{(+)} = \frac{1}{2}(F^{ij} + F^{ji})$ and $T^{(+)}_{\mu\nu} = \frac{1}{2}(T^{ij}_{\mu\nu} + T^{ji}_{\mu\nu})$ for the amplitudes,

$$F^{ij} = -i\int dx^4 e^{iK\cdot x} \langle p' | T\{D^i(\tfrac{x}{2})\, D^j(-\tfrac{x}{2})\} | p\rangle$$

and $$T^{ij}_{\mu\nu} = -i\int dx^4 e^{iK\cdot x} \langle p' | T\{A^i_\mu(\tfrac{x}{2})\, A^j_\nu(-\tfrac{x}{2})\} | p\rangle .$$

Again, in the vanishing current four-momenta limits $q, q' \to 0$ or, alternatively, $(\nu, t, w, v) \to (0,0,0,0)$ we get the relation,

$$F^{(+)}_{ij}(0,0,0,0) = i\langle p | [A^i, D^j] | p\rangle . \tag{63}$$

(The Born contribution in $q'_\mu q_\nu T^{(+)}_{\mu\nu}$ does not prevent this term from vanishing and this assertion will be explicitly shown in the following applications.) Choosing unitary indices in order to lead to experimentally accessible quantities in what follows we list the relations,[26)]

$$F^{(+)}_{12}(0,\ 0,\ 0,\ 0) = i\langle p|[A^{1+i2},\ D^{1-i2}]|p\rangle$$

so (64)

$$\langle p|\sqrt{2}\,u_0 + u_8|p\rangle = \frac{3}{2}\ \frac{F^{(+)}_{12}(0,\ 0,\ 0,\ 0)}{(\sqrt{2}+\varkappa)}\ ,$$

and

$$\frac{1}{2}\left\{F^{(+)}_{45}(0,\ 0,\ 0,\ 0) + F^{(+)}_{67}(0,\ 0,\ 0,\ 0)\right\}$$

$$= \frac{1}{2}\left\{i\langle p|[A^{4+i5}, D^{4-i5}]|p\rangle + i\langle p|[A^{6+i7}, D^{6-i7}]|p\rangle\right\}$$

so (65)

$$\langle p|\sqrt{2}\,u_0 - \frac{1}{2}u_8|p\rangle = \frac{3}{4}\cdot\frac{\left\{F^{(+)}_{45}(0,0,0,0) + F^{(+)}_{67}(0,0,0,0)\right\}}{(\sqrt{2}-\frac{\varkappa}{2})}$$

$$\equiv \frac{3}{4}\cdot\frac{F^{(+)}_{K}(0,0,0,0)}{(\sqrt{2}-\frac{\varkappa}{2})}\ .$$

If we want a proton matrix element of the linear combination of scalar densities appearing in $\theta_{00}$,

$$\langle p|u_0 + \varkappa\, u_8|p\rangle$$

(66)

$$= \frac{1}{2\sqrt{2}}\left(\frac{1+2\sqrt{2}\,\varkappa}{\sqrt{2}+\varkappa}\right)F^{(+)}_{12}(0,0,0,0) + \frac{1}{2\sqrt{2}}\left(\frac{1-\sqrt{2}\,\varkappa}{\sqrt{2}-\frac{\varkappa}{2}}\right)F^{(+)}_{K}(0,0,0,0).$$

Now the question is prompted of how the amplitudes $F^{(+)}(0,0,0,0)$ are related to hadron amplitudes at physical values of their scattering invariants. Equation (22) which was used for the pseudo-scalar scattering case no longer is applicable in general because of the presence here of Born terms and/or of unphysical threshold continuum contributions. These singularities in the energy variable $\nu$ interfere with writing the approximate statement of double pseudo-scalar pole dominance of the entire weak amplitude $F^{(+)}$. Accordingly, we apply an extrapolation procedure following Weisberger[4-b] who dealt with the unitary spin antisymmetric case, the Adler-Weisberger calculation. For the discussion of the $\bar{K}N$ channel extrapolation involved in (65), we will use a finite-energy-sum-rule approach[27] which is not so sensitive to the unphysical continuum contributions. Also in this section we will discuss more general extrapolation procedures, such as mass dispersion relations, which indicate departures from especially the simple pion or kaon pole dominance.

1) Low-Energy Extrapolations

We are first going to estimate the RHS of (64) by evaluating $F^{(+)}_{12}(0, 0, 0, 0)$ as well as using the relation for $\varkappa$, eqn. (3). At $q, q' = 0$, the Born singularity in $F^{(+)}$ consisting of the neutron, single-particle intermediate state is separated off[4-b] and the remaining piece of $F^{(+)}$ is then dominated by the double pseudo-scalar pole term (see Figure 4-a for the singularity structure),

$$F^{(+)}_{12}(0,0,0,0) = F^{(+)B}(0,0,0,0) + f_\pi^2\, \tilde{T}^{(+)}_{p\pi}(0,0,m_\pi^2,0). \qquad (67)$$

From Appendix B, $F^{(+)B}(\nu, 0, q^2, 0) = \dfrac{D^2(q^2)}{2M_p} \dfrac{[\nu^2 - \Delta M\, \nu_n]}{\nu^2 - \nu_n^2}$ so that $F^{(+)B}(0,0,0,0) = (M_p + M_n)G_A^2 = f_\pi^2 \dfrac{g^2_{\pi NN}}{M}$ using the Goldberger-Treiman relation. The "tilde" on $\widetilde{T}^{(+)}_{p\pi}$ indicates that this amplitude is the non-Born part of the pion-nucleon amplitude $T^{(+)}_{p\pi} = \frac{1}{2}(T_{p\pi^+} + T_{p\pi^-})$ which satisfies[28)]

$$T^{(+)}_{p\pi}(\nu, 0, m_\pi^2, 0) = T^{(+)B}_{p\pi}(\nu, 0, m_\pi^2, 0) + \widetilde{T}^{(+)}_{p\pi}(\nu, 0, m_\pi^2, 0)$$

$$= \frac{g^2_{\pi NN}}{M_p} \frac{[\nu^2 - \Delta M\, \nu_B]}{\nu^2 - \nu_B^2} + \widetilde{T}^{(+)}_{p\pi}(\nu, 0, m_\pi^2, 0)\,, \tag{68}$$

$$\nu_B = \frac{1}{2M_p}\,[M_n^2 - M_p^2 - m_\pi^2]\,.$$

Writing a threshold subtracted dispersion relation for $T^{(+)}_{p\pi}(\nu, 0, m_\pi^2, 0)$,

$$T^{(+)}_{\pi p}(\nu, 0, m_\pi^2, 0) = T^{(+)}_{p\pi, th} + \frac{g^2_{\pi NN}}{M} \frac{(m_\pi^2 - \nu^2)[\nu_B - \Delta M]\nu_B}{(\nu^2 - \nu_B^2)(m_\pi^2 - \nu_B^2)}$$

$$+ \frac{2(\nu^2 - m_\pi^2)}{\pi} \int_{\nu_t}^{\infty} d\nu' \, \frac{\nu' \,\mathrm{Im}\, T^{(+)}_{\pi p}(\nu', 0, m_\pi^2, 0)}{(\nu'^2 - \nu^2)(\nu'^2 - m_\pi^2)}$$

and evaluating it at $\nu = 0$,

$$\widetilde{T}^{(+)}_{\pi p}(0, 0, m_\pi^2, 0) = T^{(+)}_{p\pi, th} - \frac{g^2_{\pi NN}}{M} - \frac{2m_\pi^2}{\pi} \int_{\nu_t}^{\infty} d\nu \, \frac{\mathrm{Im}\, T^{(+)}_{\pi p}(\nu, 0, m_\pi^2, 0)}{\nu(\nu^2 - m_\pi^2)}$$

$$+ \text{ negligible terms.}$$

In this expression the threshold amplitude $T^{(+)}_{\pi p, th} \equiv T^{(+)}_{\pi p}(m_\pi, 0, m_\pi^2, 0)$. Inserting into eqn. (67),

$$f_\pi^{-2} F^{(+)}_{12}(0,0,0,0) = T^{(+)}_{p\pi, th} - \frac{2m_\pi^2}{\pi} \int_{\nu_t}^{\infty} d\nu \; \frac{\operatorname{Im} T^{(+)}_{p\pi}(\nu, 0, m_\pi^2, 0)}{\nu(\nu^2 - m_\pi^2)} . \tag{69}$$

The normalization is such that $T^{(+)}_{p\pi, th} = -4\pi(1 + \frac{m_\pi}{M}) \cdot \frac{1}{3}(a_1 + 2a_3)$, where $a_1$, $a_3$ are the s-wave $I = 1/2$, $3/2$ scattering lengths, and $\operatorname{Im} T^{(+)}_{p\pi}(\nu, 0, m_\pi^2, 0) = -\frac{1}{2}(\nu^2 - m_\pi^2)^{1/2}\{\sigma_{p\pi^+} + \sigma_{p\pi^-}\}$. Using the value[30] $(a_1 + 2a_3) = 0.069\, m_\pi^{-1}$ as well as an evaluation[31] of the integral $\simeq 4\pi(.13)m_\pi^{-1}$, one obtains $f_\pi^{-2} F^{(+)}_{12}(0,0,0,0) \approx 4\pi(.10)m_\pi^{-1}$. Using this number in eqn. (64) together with $(\sqrt{2} + \varkappa)$ $\approx \frac{3}{2}\sqrt{2}\,(m_\pi^2/m_K^2)(f_K/f_\pi)$ one receives

$$\langle p|\, u_0 + \frac{u_8}{\sqrt{2}} \,|p\rangle \approx 790 \text{ MeV}. \tag{70}$$

Consistent with our statements regarding SU(3) symmetry for pseudoscalar states and the relation (3) for $\varkappa$, SU(3) symmetric baryon states would imply $\langle p|\, \theta^x_{00} + u_0 \,|p\rangle$ = central mass of baryon octet $\simeq$ 1150 MeV and $\langle p|\, u_8 \,|p\rangle \simeq$ 150 MeV. However, we see that eqn. (64) is somewhat sensitive to the precise value for $\varkappa$ (it is the factor $(\sqrt{2} + \varkappa)^{-1}$ which, by (3), multiplies the small quantity $F^{(+)}_{12}(0,0,0,0)$ by the large ratio $\sim m_K^2/m_\pi^2$). For example, should $f_K/f_\pi$ be set equal in (3) to one or left at 1.30? The answer for the nucleon scalar density changes by 30%. Also, alternative values in the literature for the scattering lengths would revise the estimate in (70) upward by as much as 25%.

We turn now to the nucleon matrix element in (65). Whereas its determination would not be sensitive to the value for $\varkappa$, other problems will be encountered such as the approximation of kaon pole dominance.[32)] Once again we divide $F_K^{(+)}(\nu, t, w, v) \equiv \{F_{45}^{(+)} + F_{67}^{(+)}\}$ into two pieces as $q, q' \to 0$: The Born part which includes the single-particle intermediate states $\Lambda$ and $\Sigma$ and the remaining part which is dominated by the double kaon pole term,

$$F_K^{(+)}(0,0,0,0) = F_K^{(+)B}(0,0,0,0) + f_K^2 \, \tilde{T}_K^{(+)}(0,0,m_K^2,0). \qquad (71)$$

Using (B-3) and Goldberger-Treiman relations for kaons,

$$f_K^{-2} F_K^{(+)B}(0,0,0,0) = \left\{ \frac{g_{K\Lambda N}^2}{(M_\Lambda+M)} + \frac{3g_{K\Sigma N}^2}{(M_\Sigma+M)} \right\} \qquad (72)$$

where the coupling constants are normalized so that in SU(3) symmetry $g_{K\Lambda N}^2 = \frac{1}{3}(3-2\alpha)^2 g_{\pi NN}^2$ and $g_{KN\Sigma}^2 = (1-2\alpha)^2 g_{\pi NN}^2$ with $\alpha/(1-\alpha) = D/F$. A threshold subtracted representation for $T_K^{(+)}(\nu, 0, m_K^2, 0)$ might tend to emphasize the unphysical continuum contributions, especially for a complex scattering length parameterization of these $\Lambda\pi$, $\Sigma\pi$, ... channels. Alternatively, we will apply a finite-energy-sum-rule representation (Appendix C) which is not sensitive to unphysical continuum contributions, but instead involves subtracting out of the high-energy (Regge) part of the amplitude $T_K^{(+)}(\nu, 0, m_K^2, 0)$. Thus, in our expression for $\tilde{T}_K^{(+)}$, Regge pole parameters for KN, $\bar{K}N$ processes at $t = 0$ will be prominently involved. At present, there are measurements of these parameters, and in the near future more refined data will be forthcoming. From Appendix C,

$$\tilde{T}_K^{(+)}(0,0,m_K^2,0) = -\left\{\frac{g_{K\Lambda N}^2}{2M} + \frac{3g_{K\Sigma N}^2}{2M}\right\} + \frac{2}{\pi}\sum_i \frac{C_i}{\alpha_i}\left(\frac{x}{\nu_0}\right)^{\alpha_i}$$

$$+ \frac{2}{\pi}\int_{\nu_c}^{x} \frac{d\nu}{\nu}\, \mathrm{Im}\, T_K^{(+)}(\nu,0,m_K^2,0)\,. \qquad (73)$$

Inserting (73) into eqn. (71),

$$f_K^{-2}F_K^{(+)}(0,0,0,0) = -\frac{g_{K\Lambda N}^2}{2M}\left(\frac{M_\Lambda - M}{M_\Lambda + M}\right) - \frac{3g_{K\Sigma N}^2}{2M}\left(\frac{M_\Sigma - M}{M_\Sigma + M}\right)$$

$$+ \frac{2}{\pi}\sum_i \frac{C_i}{\alpha_i}\left(\frac{x}{\nu_0}\right)^{\alpha_i} + \frac{2}{\pi}\int_{\nu_c}^{x} \frac{d\nu}{\nu}\, \mathrm{Im}\, T_K^{(+)}(\nu,0,m_K^2,0). \qquad (74)$$

For normalization, in the scattering region $\mathrm{Im}\, T_K^{(+)}(\nu,0,m_K^2,0) = -\frac{1}{2}(\nu^2 - m_K^2)^{1/2}\{\sigma_{K^+p} + \sigma_{K^-p} + \sigma_{K^+n} + \sigma_{K^-n}\}$. Here $C_i$, $\alpha_i$ are couplings and trajectories for P, P' Regge poles[33]; $\nu_c$ is the continuum threshold corresponding to $s = (M_\Lambda + m_\pi)^2$; and $\nu = x$ is a cutoff for which the evaluation of (74) is found to be quite insensitive (values x corresponding to kaon lab momenta equal to 4.00, 6.00 and 8.00 BEV/c support this statement). We choose an s-wave, complex scattering length parameterization which interprets $Y_0^*$ (1405, $1/2^-$) as a $\bar{K}N$, I = 0 virtual bound-state resonance[34] and we treat $Y_1^*$ (1385, $3/2^+$) as a narrow scattering resonance with the $Y_1^* \bar{K}N$ coupling constant connected to $N^* N\pi$ in broken SU(3)[35]. We will employ both the cases of symmetric[36] and broken[35] SU(3) coupling constants $g_{K\Lambda N}$, $g_{K\Sigma N}$. The various contributions to the quantity $F_K^{(+)}(0,0,0,0)$ of eqn. (74) are now explicitly displayed,

$$
\begin{array}{llllll}
f_K^{-2} F_K^{(+)}(0,0,0,0) = & -11.6 & +6.05 & +437 & -405\ \mathrm{BEV}^{-1} & = 26.5\ \mathrm{BEV}^{-1}. \\
& (SU(3)\ \Lambda,\Sigma) & (Y_1^*) & (\text{Regge piece}) & (\text{integral}) &
\end{array}
\tag{75}
$$

Here the separate Regge and integral pieces in (75) were evaluated for the value of x corresponding to $p^K_{Lab} = 6.00$ BEV/c. If one uses the broken SU(3) values for $\Lambda, \Sigma$ couplings, the first numerical quantity in the RHS of (75) becomes $(-6.0)\,\mathrm{BEV}^{-1}$ so that $f_K^{-2} F_K^{(+)}(0,0,0,0) = 32.1\ \mathrm{BEV}^{-1}$. Using (75) with $f_K \simeq 1.30\, f_\pi$,

$$F_K^{(+)}(0,0,0,0) \simeq .72\ \mathrm{BEV}.$$

Inserting this value into eqn. (65),

$$\langle p|\, u_0 - \frac{1}{2\sqrt{2}}\, u_8 \,|p\rangle \simeq 180\ \mathrm{MEV}. \tag{76}$$

This matrix element is small relative to the central octet baryon mass $\simeq$ 1150 MEV. Since, as was mentioned before, $\langle p|\,u_8\,|p\rangle$ is only $\simeq$ 150 MEV, comparison with the larger value (70) suggests that with these extrapolation techniques a definite conclusion cannot be reached on the size of the nucleon scalar density terms.[37]

We now turn to a critical discussion of some generalizations of the low-energy extrapolation.

2) Discussion of Generalizations

Since the (non-Born) divergence scattering of nucleons was approximated by double pseudo-scalar pole dominance at $q^2 = 0$, the first generalization of this circumstance will involve multiple meson production and scattering amplitudes. However, even this modest extension would involve sketchy experimental quantities and somewhat arduous theoretical development. We content ourselves with a formal expansion for $F^{(+)}(0,0,0,0)$ into pseudo-scalar "resonance"-proton scattering amplitudes with weak proportionality constants representing resonance couplings to the current divergences. The form of the expansion is just what one would expect; yet the means for obtaining it, a mass dispersion relation, makes rather clear the nature of the simplifications involved in obtaining, say, eqn. (71).

Once again we want to connect $F^{(+)}(\nu, q^2) \equiv F^{(+)}(\nu, 0, q^2, 0) = \frac{1}{2}\{F^{ij} + F^{ji}\}$ at $q = 0$ with hadronic amplitudes and, thence, to the scalar density terms via eqn. (63),

$$F^{(+)}(0,0,0,0) = i\,\langle p|\,[A^+,\ D^-]\,|p\rangle, \quad "\pm" = i \pm (i)j.$$

The amplitude $F^{(+)}(\nu, q^2)$ is divided into two pieces for use at $q = 0$,

$$F^{(+)}(\nu, q^2) = F^{(+)\,B}(\nu, q^2) + \tilde{F}^{(+)}(\nu, q^2)\,. \tag{77}$$

The first, Born piece is calculated as before from perturbation theory or from writing unsubtracted dispersion relations in $\nu$ (for fixed $q^2$) for the component invariant amplitudes using the one-particle intermediate state imaginary part.[28)] The second, "non-Born" piece is assumed to satisfy an unsubtracted dispersion relation in $q^2$ (for fixed $\nu$) using the imaginary part given directly from (77) as

$\tilde{f}^{(+)} = f^{(+)} - f^{(+)B}$, where[38)]

$$f^{(+)}(\nu, q^2) = -\frac{1}{4}\int dx^4\, e^{iq\cdot x}\langle p|\left\{[D^+(x), D^-(0)] + [D^-(x), D^+(0)]\right\}|p\rangle$$
$$= -\frac{1}{4}\sum_n (2\pi)^4 \delta(p+q-p_n)\langle p|D^+(0)|n\rangle\langle n|D^-(0)|p\rangle + \text{other terms} \tag{78}$$

and $f^{(+)B}$ just involves the one-particle intermediate state in the sum. Therefore,

$$F^{(+)B}(0,0) + \frac{1}{\pi}\int \frac{dq^2}{q^2}\tilde{f}^{(+)}(q^2, 0) = i\langle p|[A^+, D^-]|p\rangle\ . \tag{79}$$

Taking the crossing symmetry property of $f^{(+)}(q^2, \nu)$ into account, (79) was written for $\nu > 0$ and then $\nu \to 0^+$ was taken; otherwise, a minus sign would multiply the integral appearing in (79) for $\nu < 0$. From Appendix D, we then take account in (78) of hadron production disconnected matrix elements and write for these in the resonance approximation the following decomposition,

$$\langle \alpha p|D|p\rangle = \langle p|p\rangle\langle\alpha|D|0\rangle + \sum_\beta{}' \frac{T(p\alpha;\, p\beta)\ \langle\beta|D|0\rangle}{q^2 - m_\beta^2 + i\epsilon}, \tag{80}$$

$\alpha, \beta$ denote $J^P = 0^-$ pseudo-scalar "resonances".

We then insert such expressions into (78) and cancel out the undesirable "double-pinch" terms using the unitarity condition for $T(p\alpha;\, p\beta)$ (see Appendix D). The integration over $\tilde{f}(q^2, \nu)$ in eqn. (79) then "picks off" the double pole residues which are strong amplitudes multiplied by weak proportionality factors (see Figure 5 for an illustration of this circumstance):

$$F^{(+)B}(0,0) + \frac{1}{2}\sum_{\alpha,\beta} f_\alpha f_\beta \left\{ \tilde{T}(p\alpha^+;\, p\beta^+) + \tilde{T}(p\alpha^-;\, p\beta^-) \right\} = i\langle p|[A^+, D^-]|p\rangle \tag{79$'$}$$

$$= \text{scalar density terms.}$$

Here the amplitudes $\tilde{T}(p\alpha^{\pm};\, p\beta^{\pm})$ which describe the strong process $\beta^{\pm} + p \to \alpha^{\pm} + p$ are evaluated for $\nu = 0$ and the weak proportionality factors are defined from $\langle 0|D^+|\alpha^-\rangle = m_\alpha^2 f_\alpha$. The case, for example, in which "$\pm$" $= 1 \pm i2$ and $\alpha = \beta = \pi$ immediately gives (67).

The actual form of (79)' does not constitute any practical improvement over (67) or (71), nor does it treat all possible (disconnected) graph corrections. An example of a representation which at least exhibits all possible disconnected contributions is given by the simple expansion[38)] (which is equivalent to writing unsubtracted dispersion relations in $q_0$),

$$F^{ij}(\nu, q^2) = -i\int dx^4\, e^{iq\cdot x}\, \langle p|\theta(x_0)[D^i(x), D^j(0)]|p\rangle$$

$$= \frac{1}{\pi}\int \frac{dq_0'}{q_0' - q_0 - i\epsilon}\, f^{ij}$$

where $$f^{ij} = -\frac{1}{2}\int dx^4\, e^{iq\cdot x}\, \langle p|[D^i(x), D^j(0)]|p\rangle \tag{81}$$

$$= -\frac{1}{2}\sum_n (2\pi)^4\,\delta(p+q-p_n)\,\langle p|D^i|n\rangle_c\,\langle n|D^j|p\rangle_c$$

$$-\frac{1}{2}\sum_\alpha (2\pi)^4\,\delta(q-p_\alpha)\,\langle 0|D^i|\alpha\rangle\,\langle \alpha p|D^j|p\rangle_c$$

$$- \frac{1}{2} \sum_{\beta} (2\pi)^4 \delta(q - p_\beta) \langle p | D^i | p\beta \rangle_c \langle \beta | D^j | 0 \rangle$$

$$- \frac{1}{2} \sum_{\overline{m}} (2\pi)^4 \delta(p - q + p_{\overline{m}}) \langle 0 | D^i | p\overline{m} \rangle \langle p\overline{m} | D^j | 0 \rangle$$

$$+ \text{ (crossed terms, } q \leftrightarrow -q \text{ and } i \leftrightarrow j \text{ ) .}$$

Here $\langle p | D^i | n \rangle_c$ denotes the "connected part" of $\langle p | D^i | n \rangle$ and the new sum represents the "z-graph" contribution due to the disconnected matrix element in the decomposition $\langle p | D^i | pp\overline{m} \rangle = \langle p | p \rangle \langle 0 | D^i | p\overline{m} \rangle + \langle p | D^i | pp\overline{m} \rangle_c$. We will now discuss the Breit frame specialization of this formula in which Fubini and Furlan[41)] have derived an equation relating the scalar density term to amplitudes together with corrections (still within the framework of an extrapolation from $q = 0$). However, as we shall indicate, the method in practice is inapplicable for estimating scalar density terms because of, among other things, the presence of an arbitrary subtraction constant at $q_0 \to \infty$.[41)]

We sketch briefly the derivation of the Fubini, Furlan equation. One writes the current identity (62) with forward kinematics in the Breit frame of the target proton so that $p = (M, \underset{\sim}{0})$. Then one takes $\underset{\sim}{q} = 0$ so that $\nu = q_0$ and $q^2 = q_0^2$ and there results a kinematical restriction, the parabola $q^2 = \nu^2$ (see Figure 4-b). From the identity, $q_0 \to 0$ gives the condition $F^{(+)}(0,0) = i\langle p | [A^+, D^-] | p \rangle$. From the <u>unsubtracted</u> representation (81), the $q_0 \to \infty$ limit gives

$$F^{(+)}(q_0 \to \infty) \to \frac{c^{(+)}}{q_0^2} , \quad c^{(+)} = \frac{i}{2} \int dx^3 \langle p | \Big\{ [\dot{D}^+(0, \underset{\sim}{x}), D^-(0)] + [\dot{D}^-(0, \underset{\sim}{x}), D^+(0)] \Big\} | p \rangle . \qquad (82)$$

Now if one considers the quantity $(1 - \frac{q_0^2}{m^2})^2 \, F^{(+)}(q_0)$ in the limit for which $q_0 \to m$ ($m$ is the pseudo-scalar meson mass like $m_\pi$ or $m_K$), then along the parabola $\nu \to m$, $q^2 \to m^2$ and we get from the disconnected meson terms in expansion (81),

$$\mathcal{F}^{(+)}(q_0) \equiv (1 - \frac{q_0^2}{m_\pi^2})^2 \, F^{(+)}(q_0) \underset{q_0 \to m_\pi}{\to} f_\pi^2 \, T^{(+)}_{p\pi, th} \, . \tag{83}$$

(Take the $\pi$ symbol for definiteness.)

Now one defines the function $G(q_0) = \mathcal{F}^{(+)}(q_0)/(q_0^2 - m_\pi^2)$ for which a Cauchy integral is written. By construction the function $G$ satisfies,

$$G \to \frac{c^{(+)}}{m_\pi^4} \, , \quad q_0 \to \infty \; ;$$

$$G \to -i \langle p|[A^+, D^-]|p\rangle / m_\pi^2 , \quad q_0 \to 0 \; ;$$

and

$$G \to f_\pi^2 \frac{T^{(+)}_{p\pi, th}}{(q_0^2 - m_\pi^2)} \, , \quad q_0 \to \pm \, m_\pi \, .$$

Therefore, $$G(q_0) = G(q_0 = \infty) + (\text{poles}) + \frac{1}{\pi} \int \frac{dq_0'}{q_0' - q_0} \, \operatorname{Im} G(q_0') \tag{84}$$

and $$i\langle p|[A^+, D^-]|p\rangle = f_\pi^2 \, T^{(+)}_{\pi p, th} - \frac{2m_\pi^2}{\pi} \int_{m_\pi}^{\infty} dq_0 \frac{\operatorname{Im} \mathcal{F}^{(+)}(q_0)}{q_0 (q_0^2 - m_\pi^2)} - \frac{c^{(+)}}{m_\pi^2} \, .$$

Note the resemblance of the two formulas (84) and (69); however, it is only when $q_0$ is near $m_\pi$ that the imaginary part in (84) describes $\pi p$ scattering. Also there appears in (84) the quantity $c^{(+)}$ which is a subtraction constant at $q_0 \to \infty$ (and $\nu = q_0 = \infty$, $q^2 = q_0^2 = \infty$). To use (84), Fubini and Furlan assume that $c^{(+)}$ is zero on the basis of the argument that when $D(x) = m_\pi^2 f_\pi \phi_\pi(x)$ is used, the commutator in (82) is given by $[\dot{D}(0,\underset{\sim}{x}), D(0)] \sim [\dot{\phi}(0,\underset{\sim}{x}), \phi(0)] = (\text{c-number})\,\delta(\underset{\sim}{x})$.[38] However, any such PCAC operator statement is motivated where the application involves a small momentum transfer to the axial divergence sandwiched between states; i.e., pseudo-scalar pole dominance. Now we may take the equal-time commutator contribution to $c^{(+)}$ in the Breit frame ($\underset{\sim}{p} = 0$, $\underset{\sim}{q} = 0$) and expand:

$$\int dx^3 \langle p|[\dot{D}^+(0,\underset{\sim}{x}),\ D^-(0)]|\,p\rangle =$$

$$- i \sum_n (2\pi)^3 \delta(\underset{\sim}{p}_n)(p_n^0 - M) \langle p|D^+(0)|n\rangle\langle n|D^-(0)|\,p\rangle + \text{other term.}$$

We see that the momentum transfer in the matrix element $\langle p|D^+|n\rangle$ is given by $(p - p_n)^2 = (M - M_n)^2$ which, for any arbitrary state n, certainly need not be vanishing. In other words, with the representation (84) one must make an assumption about the equal-time commutator which defines $c^{(+)}$ in order to estimate another, equally unknown equal-time commutator $i\langle p|[A^+, D^+]|\,p\rangle$ - an unsatisfactory circumstance. Suppose, however, one were to ignore the constant $c^{(+)}$ and instead concentrated on evaluating the integral in (84). Then in this $\underset{\sim}{p} = 0$, $\underset{\sim}{q} = 0$ system, $J^P = 1/2^-$ isobars contribute from the connected matrix element sum piece in (81). Unfortunately for the $\pi N$ case, the first candidate is $N^*(1570,\ 1/2^-)$ and this corresponds to $q_0 = M_* - M \simeq 4m_\pi$ so that the parabolic restriction gives

$q^2 = 16m_\pi^2$ (not $q^2 \simeq m_\pi^2$). Then the imaginary part $\mathrm{Im}\,\mathcal{F}^{(+)}(q_0) = (1 - \frac{q_0^2}{m_\pi^2})^2 f^{(+)}(q_0)$ cannot be approximated by calculable $\pi p$ scattering. In the case with kaon-nucleon scattering quantum numbers (65) and (84) would give,

$$\frac{4}{3}(\sqrt{2} - \frac{\kappa}{2})\langle p|\sqrt{2}u_0 - \frac{1}{2}u_8|p\rangle = f_K^2 \cdot -\pi(1 + \frac{m_K}{M})\{(3a_1+a_0)+(3a_1'+a_0')\}$$

$$- \frac{2m_K^2}{\pi}\int^\infty dq_0 \frac{\mathrm{Im}\,\mathcal{F}_K^{(+)}(q_0)}{q_0(q_0^2 - m_K^2)} - \frac{c_K^{(+)}}{m_K^2}\,. \tag{85}$$

The first term on the RHS, $f_K^2\, T_{K,th}^{(+)}$, involves $I = 0, 1$ s-wave scattering lengths $a_{0,1}$ for $\bar{K}N$ scattering[34b)] and $a_{0,1}'$ for KN scattering[42)]. This threshold amplitude alone would give $\langle p|u_0 - (1/2\sqrt{2})u_8|p\rangle \simeq 450$ MEV. A possible isobar here is $Y_0^*(1405, 1/2^-)$ whose position corresponds to $q_0 = M_* - M \approx m_K$ so that $\mathrm{Im}\,\mathcal{F}_K^{(+)}(q_0) \simeq f_K^2\, \mathrm{Im}\,T_K^{(+)}(\nu)$ for a region of $q_0$. But the integral is prohibitively sensitive to an s-wave, complex scattering length parameterization of $\mathrm{Im}\,T_K^{(+)}(\nu)$ in the unphysical continuum and low-energy regions due to the threshold factor $(\nu^2 - m_K^2)^{-1}$.[43)]

From this discussion of possible estimatory extensions, one must settle, for the present, for the (inconclusive) nucleon scalar term estimates (70) and (76).

## VI. CONCLUSION

From the discussion of various pseudo-scalar meson processes such as $\pi K$ and KK low-energy scattering we have seen that it is possible to estimate the meson scalar density terms; e.g., $\langle \pi^i | u_0 + \varkappa u_8 | \pi^i \rangle = m_\pi^2$. Estimation of nucleon scalar density terms, however, contains certain ambiguities such as sensitivity to the symmetry-breaking parameter $\varkappa$ or delicacies in the extrapolation methods. It may be that a final formulation relating these terms to weak amplitudes would involve an understanding of current-induced, hadron production mechanisms for both very large energies and current masses.

# APPENDIX A

## AMPLITUDE SYMMETRY PROPERTIES IN MANDELSTAM INVARIANTS

An example is given in which simple Bose-statistics rules in the amplitude t-channel (say) are illustrated by t-channel crossing symmetry expressed in a definite reduction representation.

The kinematics of the s-channel, elastic meson scattering amplitude is given in Figure 1. The indices $\alpha, \beta, \gamma, \delta$ for the meson process $M(\alpha, q) + M(\gamma, p) \to M(\beta, q') + M(\delta, p')$ are physical particle indices in a SU(3) basis; e.g., in $K^+\pi^+$ scattering we have $\alpha = 1+i2/\sqrt{2}$, $\beta = 1-i2/\sqrt{2}$, $\gamma = 4+i5/\sqrt{2}$, $\delta = 4-i5/\sqrt{2}$. By a simple (LSZ) reduction procedure the S-matrix can be expressed as,

$$S = \langle \delta p', \beta q' | \gamma p, \alpha q \rangle = \frac{-i(2\pi)^4 \delta(p'+q'-p-q)}{(2\pi)^6 \sqrt{16 p_0' p_0 q_0' q_0}} A^{\delta\gamma}(s,t,u;q^2,q^{2\prime},p^2,p^{2\prime})$$

$$= \frac{-i(2\pi)^4 \delta(p'+q'-p-q)}{(2\pi)^6 \sqrt{16 p_0' p_0 q_0' q_0}} A^{\beta\alpha}(s,t,u;q^2,q^{2\prime},p^2,p^{2\prime})$$

where for brevity we write,

$$A^{\delta\gamma} = \int dx^4 e^{i\cdot\frac{1}{2}(p+p')x} \langle q' | -i\theta(x_0)[j^\delta(\tfrac{x}{2}), j^\gamma(-\tfrac{x}{2})] | q \rangle$$

$$\text{and} \qquad A^{\beta\alpha} = \int dx^4 e^{i\cdot\frac{1}{2}(q+q')x} \langle p' | -i\theta(x_0)[j^\beta(\tfrac{x}{2}), j^\alpha(-\tfrac{x}{2})] | p \rangle \tag{A1}$$

with the source currents satisfying $(\Box_x^2 + m_\alpha^2)\phi_\alpha(x) = j^\alpha(x)$. We define the definite t-channel charge conjugation combinations $(\mathcal{C} = \pm)$ by $A^{(\pm)}_{\delta\gamma} = \frac{1}{2}(A^{\delta\gamma,\alpha\beta} \pm A^{\delta\gamma,\beta\alpha})$ and $A^{(\pm)}_{\beta\alpha} = \frac{1}{2}(A^{\gamma\delta,\beta\alpha} \pm A^{\delta\gamma,\beta\alpha})$. Remembering how the indices are defined for elastic, s-channel scattering, namely in the "raising" and "lowering" basis, $A^{\delta\gamma,\alpha\beta} = A^{\gamma\delta,\beta\alpha}$ by charge conjugation invariance. So we will utilize the quantities $A^{(\pm)} = \frac{1}{2}(A^{\alpha\beta} \pm A^{\beta\alpha}) = \frac{1}{2}(A^{\gamma\delta} \pm A^{\delta\gamma})$.

From (1),

$$A^{\beta\alpha^*}(p', q'; p, q) = \int dx^4 e^{-i\frac{1}{2}(q+q')x} \langle p| -i\theta(x_0)[j^\alpha(\tfrac{x}{2}), j^\beta(-\tfrac{x}{2})]|p'\rangle$$

where $(j^\alpha)^* = j^\beta$ has been used.

Since, $A^{\beta\alpha^*}(p, -q'; p', -q) = \int dx^4 e^{i\cdot\frac{1}{2}(q+q')\cdot x} \langle p'| -i\theta(x_0)[j^\alpha(\tfrac{x}{2}), j^\beta(-\tfrac{x}{2})]|p\rangle$,

$$A^{\alpha\beta}(p', q'; p, q) = A^{\beta\alpha^*}(p, -q'; p', -q)$$

and (A-2)

$$A^{\gamma\delta}(p', q'; p, q) = A^{\delta\gamma^*}(-p', q; -p, q') \; .$$

Using the relations $s = (p+q)^2 = (p'+q')^2$, $u = (p-q')^2 = (p'-q)^2$ and $t = (p'-p)^2 = (q'-q)^2$ we finally obtain,

$$\begin{aligned} A^{(\pm)}(s, t, u; q^2, q^{2\prime}, p^2, p^{2\prime}) &= \pm A^{(\pm)^*}(u, t, s; q^{2\prime}, q^2, p^2, p^{2\prime}) \\ &= \pm A^{(\pm)^*}(u, t, s; q^2, q^{2\prime}, p^{2\prime}, p^2). \end{aligned} \tag{A-3}$$

These t-channel crossing relations were used in the $K\pi$ scattering length discussion, equation (17) of Section III.

Now, looking at the scattering process in the t-channel for which there are definite $\mathcal{C} = \pm$ states, application of Bose statistics simply interchanges the "initial" particles $(p', \delta) \leftrightarrow (-p, \gamma)$ as well as (if desired) the "final" ones $(-q', \beta) \leftrightarrow (q, \alpha)$. The symmetry of the amplitude under subsequent changes of Mandelstam invariants is completely equivalent to the content of equations (A-2) and (A-3).

One may also formalize PT invariance using the representation (A1). For example, the initial and final pairs of states in the s-channel may be switched so that the invariants behave as $(s, t, u; q^2, q^{2\prime}, p^2, p^{2\prime}) \rightarrow (s, t, u; q^{2\prime}, q^2, p^{2\prime}, p^2)$.

# APPENDIX B

## WEAK BORN TERM CONTRIBUTIONS

A Born term expression will be given for which the limit $q', q \to 0$ is unambiguous in the current identity (62). In particular, it will be shown that the complete Born term quantity

$$B^{(+)}(\nu, q^2) = F^{(+)B}(\nu, 0, q^2, 0) - q^\mu q^\nu T^{(+)B}_{\mu\nu}(\nu, 0, q^2, 0) \qquad \text{(B-1)}$$

satisfies the $(q, q' \to 0)$ limit,

$$B^{(+)}(0,0) = F^{(+)B}(0,0,0,0) \ . \qquad \text{(B-2)}$$

For simplicity, forward kinematics $(t = 0)$ is taken with equal mass currents $(q^2 = q^{2\prime}$ or $w = q^2$, $v = 0)$. For definiteness, the one neutron intermediate state is considered in $F^{(+)}_{12}$ and $T^{(+)}_{\mu\nu, 12}$ although one could consider (say) the lambda intermediate state in $F^{(+)}_{45}$ and $T^{(+)}_{\mu\nu, 45}$, etc.

Using the vertex $\langle p | D^{1+i2} | n\rangle = i\bar{u}(p)\gamma_5 u(n) \cdot D(q^2)$ with $D(q^2) = (M_p + M_n)F_A(q^2) + q^2 F_P(q^2)$ in perturbation theory, the Born contribution to the divergence-proton scattering amplitude $F^{(+)}_{12}(\nu, 0, q^2, 0)$ is

$$F^{(+)B}_{12}(\nu, 0, q^2, 0) = \frac{D^2(q^2)}{2M_p} \cdot \frac{[\nu^2 - \Delta M \nu_n]}{[\nu^2 - \nu_n^2]}$$

$$\Delta M = M_n - M_p, \ \nu_n = \frac{1}{2M_p}[M_n^2 - M_p^2 - q^2] \ . \qquad \text{(B-3)}$$

Similarly, with the vertex $\langle p|A_\mu^{1+i2}|n\rangle = \bar{u}(p)[\gamma_\mu F_A(q^2) + q_\mu F_P(q^2)]\gamma_5 u(n)$ in perturbation theory, the Born contribution to the axial current-proton scattering amplitude $T^{(+)}_{\mu\nu,12}(\nu, 0, q^2, 0)$ is

$$T^{(+)B}_{\mu\nu,12}(\nu, 0, q^2, 0) = \frac{1}{\nu^2 - \nu_n^2}\Big\{\frac{p_\mu p_\nu}{M^2}\cdot \nu_n F_A^2 + \Big(\frac{p_\mu}{M} q_\nu + \frac{p_\nu}{M} q_\mu\Big)[F_A^2 + \Delta M F_A F_P]\frac{\nu}{2M}$$

$$+ q_\mu q_\nu \cdot \frac{1}{2M}[(\nu^2 - \Delta M \nu_n)F_P^2 - 2F_A F_P \nu_n] + g_{\mu\nu}\cdot -[\nu^2 + (M_n + M)\nu_n]\frac{F_A^2}{2M}\Big\}. \tag{B-4}$$

Now, just computing the quantity $q^\mu q^\nu T^{(+)B}_{\mu\nu,12}$,

$$q^\mu q^\nu T^{(+)B}_{\mu\nu,12} = (M + M_n)F_A^2 - 2F_A D + F_{12}^{(+)B}$$

so that,

$$B_{12}^{(+)}(\nu, q^2) \equiv F_{12}^{(+)B} - q^\mu q^\nu T^{(+)B}_{\mu\nu,12}$$

$$= 2F_A(q^2)D(q^2) - (M + M_n)F_A^2(q^2)\,. \tag{B-5}$$

When $q \to 0$ or $(\nu, q^2) = (0, 0)$,

$$B_{12}^{(+)}(0, 0) = (M + M_n)G_A^2\,. \tag{B-6}$$

This equation alone would define the Born-singularity separation used in obtaining (67) from (62) in Section V. From the expression for $F_{12}^{(+)B}(\nu, 0, q^2, 0)$ in (B-3) we see also that when $\nu, q^2 \to 0$ (any order),

$$B_{12}^{(+)}(0,0) = F_{12}^{(+)B}(0,0,0,0) .$$

No generality was lost in this last relation by saying $\Delta M = M_n - M_p$ (or $M_\Lambda - M_p$, etc.) is small but non-zero in the precise limiting process $q \rightarrow 0$.

# APPENDIX C

## FINITE ENERGY SUM RULE REPRESENTATION

In the text we deal with the strong amplitude $T_K^{(+)}(\nu, 0, m_K^2, 0) = \frac{1}{2}\left\{T_{K^+p} + T_{K^-p} + T_{K^+n} + T_{K^-n}\right\}$ to which the even charge-conjugation Regge poles P, P' contribute at high energy according to

$$T_K^{(+)}(\nu, 0, m_K^2, 0) \underset{\nu \to \infty}{\to} \sum_i C_i \cdot \frac{e^{-i\pi\alpha_i} + 1}{\sin \pi \alpha_i} \left(\frac{\nu}{\nu_0}\right)^{\alpha_i} , \qquad i = P, P' . \tag{C-1}$$

To derive the finite-energy-sum-rule (73) we first define a new function in terms of $T_K^{(+)}(\nu)$ which will have a better asymptotic behavior in $\nu$:

$$H^{(+)}(\nu) = T_K^{(+)}(\nu) - \sum_i C_i \cdot \frac{e^{-i\pi\alpha_i} + 1}{\sin \pi \alpha_i} \left(\frac{\nu}{\nu_0}\right)^{\alpha_i} . \tag{C-2}$$

In this expression, all the known Regge trajectories P, P' have been subtracted out so that $H(\nu) \to \nu^{\delta}$ , $\delta < 0$ when $\nu \to \infty$ . We now write a convergent, unsubtracted dispersion relation for $H^{(+)}(\nu)$,

$$\operatorname{Re} H^{(+)}(\nu) = \frac{2}{\pi} \int_0^\infty d\nu' \, \frac{\nu' \operatorname{Im} H^{(+)}(\nu')}{\nu'^2 - \nu^2} .$$

Re-expressing this dispersion relation in terms of $T_K^{(+)}(\nu)$ as well as the Regge parameters and extracting the single particle $\Lambda, \Sigma$ integral contributions,

$$\mathrm{Re}\, T_K^{(+)}(\nu) - \sum_i C_i \cdot \frac{e^{-i\pi\alpha_i}+1}{\sin \pi \alpha_i} \left(\frac{\nu}{\nu_0}\right)^{\alpha_i} = \frac{g_{K\Lambda N}^2}{2M} \frac{[\nu_\Lambda - (M_\Lambda - M)]\nu_\Lambda}{\nu^2 - \nu_\Lambda^2}$$

$$+ \frac{3g_{K\Sigma N}^2}{2M} \frac{[\nu_\Sigma - (M_\Sigma - M)]\nu_\Sigma}{\nu^2 - \nu_\Sigma^2} + \frac{2}{\pi} \int^\infty d\nu' \frac{\nu'}{\nu'^2 - \nu^2} \left\{ \mathrm{Im}\, T_K^{(+)}(\nu') \right. \quad \text{(C-3)}$$

$$\left. + \sum_i C_i \left(\frac{\nu'}{\nu_0}\right)^{\alpha_i} \right\} .$$

The amplitude $\tilde{T}_K^{(+)}$ is defined from $T_K^{(+)} = T_K^{(+)B} + \tilde{T}_K^{(+)}$, where the Born term is given by[28)]

$$T_K^{(+)B}(\nu, 0, m_K^2, 0) = \frac{g_{K\Lambda N}^2}{2M} \frac{[\nu^2 - (M_\Lambda - M)\nu_\Lambda]}{\nu^2 - \nu_\Lambda^2}$$

$$+ \frac{3g_{K\Sigma N}^2}{2M} \frac{[\nu^2 - (M_\Sigma - M)\nu_\Sigma]}{\nu^2 - \nu_\Sigma^2}, \quad \nu_{\Lambda,\Sigma} = \frac{1}{2M}[M_{\Lambda,\Sigma}^2 - M^2 - m_K^2] . \quad \text{(C-4)}$$

Let us cut off the integral in (C-3) at $\nu' = x$, the onset of the Regge region where the absorptive part vanishes by (C-1), and denote by $\nu' = \nu_c$ the unphysical continuum threshold in $\mathrm{Im}\, T_K^{(+)}$. Then setting $\nu = 0$ in (C-3) and (C-4) (and remembering that the integral in (C-3) extends now from $\nu' = 0$ to $\nu' = x$ for the Regge part),

$$\tilde{T}_K^{(+)}(0,0,m_K^2,0) = -\left\{ \frac{g_{K\Lambda N}^2}{2M} + \frac{3g_{K\Sigma N}^2}{2M} \right\} + \frac{2}{\pi} \sum_i \frac{C_i}{\alpha_i} \left( \frac{x}{\nu_0} \right)^{\alpha_i}$$
$$+ \frac{2}{\pi} \int_{\nu_c}^{x} \frac{d\nu}{\nu} \, \mathrm{Im}\, T_K^{(+)}(\nu,0,m_K^2,0) \quad . \tag{73}$$

# APPENDIX D

## SOME DETAILS OF MASS DISPERSION INTEGRATION

In this Appendix we give some of the details involved in evaluating the mass dispersion integral in (79) so as to arrive at the expansion in (79)'.

Among the general states appearing in the sum over intermediate states (78), consider the state $|p\alpha\rangle$, where $\alpha$ is a $J^P = 0^-$ system which we approximate here as a resonance. Then, using the LSZ reduction technique, we may decompose the matrix element $\langle p|D^+|p\alpha^-\rangle$ into a connected and a disconnected piece,

$$\langle p|D^+|\alpha^- p, \text{out}\rangle = \langle 0|b_{\text{out}}(p)D^+(0)|\alpha^- p, \text{out}\rangle$$

$$= \langle 0|D^+(0)b_{\text{out}}(p)|\alpha^- p, \text{out}\rangle + \langle 0|[b_{\text{out}}(p), D^+(0)]|\alpha^- p, \text{out}\rangle$$

$$= \langle p|p\rangle\langle 0|D^+|\alpha^-\rangle - i\int dx^4 e^{ip\cdot x}\bar{u}(p,s)(i\gamma\cdot\overrightarrow{\nabla} - M)\langle 0|T\{\psi(x)D^+(0)\}|p\alpha^-, \text{out}\rangle. \tag{D-1}$$

For the connected pieces, an unsubtracted dispersion relation in $q^2$ is written[39)] with the imaginary part approximated by narrow $J^P = 0^-$ resonances (so we shall neglect anomalous singularity effects); e.g.,

$$\langle 0|[b_{out}(p), D^{+}(0)]|\alpha^{-} p, out\rangle$$

$$= \frac{1}{\pi}\int \frac{dk^2}{k^2 - q^2 + i\epsilon}\left\{-\pi \sum_{\beta} \delta(k^2 - m_\beta^2) T^*(p\alpha^-; p\beta^-)\langle 0|D^+|\beta^-\rangle\right\}$$

$$= \sum_{\beta} \frac{T^*(p\alpha^-; p\beta^-)\langle 0|D^+|\beta^-\rangle}{q^2 - m_\beta^2 - i\epsilon} \quad . \qquad \text{(D-2)}$$

Similarly, the connected part of $\langle\alpha^- p, out|D^-(0)|p\rangle$ is given by,

$$\langle\alpha^- p, out|[D^-(0), b^+(p)]|0\rangle = \sum_{\beta} \frac{T(p\alpha^-; p\beta^-)\langle\beta^-|D^-|0\rangle}{q^2 - m_\beta^2 + i\epsilon} \quad . \qquad \text{(D-3)}$$

Inserting the matrix element (D-1) into the expression for $\tilde{f}^{(+)}(\nu, q^2)$ and remembering that we have subtracted off the completely disconnected vacuum graphs,[38]

$$(-4)\tilde{f}^{(+)}(\nu, q^2) = \sum_{\alpha,\beta} (2\pi)^4 \langle 0|D^+|\alpha^-\rangle\langle\beta^-|D^-|0\rangle\left\{\frac{T(p\alpha^-; p\beta^-)\delta(q_0 - p_{\alpha 0})}{q^2 - m_\beta^2 + i\epsilon}\right.$$

$$\left. + \frac{T^*(p\beta^-; p\alpha^-)\delta(q_0 - p_{\beta 0})}{q^2 - m_\alpha^2 - i\epsilon}\right\} + \sum_{\alpha,\beta} \langle 0|D^+|\alpha^-\rangle\langle\beta^-|D^-|0\rangle \qquad \text{(D-4)}$$

$$\frac{\left\{\sum_n{}' (2\pi)^4 \delta(p+q-p_n) T^*(n; p\alpha^-) T(n; p\beta^-)\right\}}{(q^2 - m_\alpha^2 - i\epsilon)(q^2 - m_\beta^2 + i\epsilon)} + \text{other, similar terms.}$$

The prime on the sum above the "double-pinch" poles denotes that the one-particle intermediate state is not included. We now make use in (D-4) of the relation

$$\delta(q_0 - p_{\alpha 0}) = 2p_{\alpha 0} \cdot \frac{1}{2\pi i} \left\{ \frac{1}{q^2 - m_\alpha^2 - i\epsilon} - \frac{1}{q^2 - m_\alpha^2 + i\epsilon} \right\}$$

and account for all flux normalization factors. Then, keeping track of the one-particle state, a cancellation takes place in (D-4) which removes the second term (a continued unitarity relation) and part of the first term since

$$\frac{\frac{1}{i}\left\{T(p\alpha^-; p\beta^-) - T^*(p\beta^-; p\alpha^-)\right\}}{(q^2 - m_\alpha^2 - i\epsilon)(q^2 - m_\beta^2 + i\epsilon)} = \frac{-\sum_n' (2\pi)^4\, \delta(p+q-p_n)\, T^*(n; p\alpha^-) T(n; p\beta^-)}{(q^2 - m_\alpha^2 - i\epsilon)\,(q^2 - m_\beta^2 + i\epsilon)}.$$

Thus one receives[40] $(\nu > 0)$,

$$\tilde{f}^{(+)}(\nu, q^2) = \frac{1}{4i} \sum_{\alpha,\beta} \langle 0|D^+|\alpha^-\rangle\langle\beta^-|D^-|0\rangle \left\{ \frac{\tilde{T}(p\alpha^-; p\beta^-)}{(q^2 - m_\alpha^2 + i\epsilon)(q^2 - m_\beta^2 + i\epsilon)} \right.$$

$$\left. - \frac{\tilde{T}^*(p\beta^-; p\alpha^-)}{(q^2 - m_\alpha^2 - i\epsilon)(q^2 - m_\beta^2 - i\epsilon)} \right\} + (\text{term with } + \leftrightarrow -). \qquad \text{(D-5)}$$

The contour of the integral $\frac{1}{\pi}\int \frac{dq^2}{q^2}\, \tilde{f}^{(+)}(\nu, q^2)$ is made to run not directly along the real $q^2$ axis, but rather along a series of circles which surround the singularities at $q^2 = m_\alpha^2, m_\beta^2$ (recall that

$\tilde{T}^*(p\beta;\, p\alpha)$ is the boundary value of $\tilde{T}(p\alpha;\, p\beta)$ approaching from below the real axis). Then,

$$\frac{1}{\pi}\int \frac{dq^2}{q^2}\,\tilde{f}^{(+)}(q^2,\nu) = \frac{1}{2}\sum_{\alpha,\beta} \frac{\langle 0|D^+|\alpha^-\rangle}{m_\alpha^2}\,\frac{\langle \beta^-|D^-|0\rangle}{m_\beta^2}\left\{\tilde{T}(p\alpha^-;\, p\beta^-) + \tilde{T}(p\alpha^+;\, p\beta^+)\right\}$$

leads to equation (79)' with the definitions $\langle 0|D^+|\alpha^-\rangle = m_\alpha^2 f_\alpha$ .

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2. M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

3. J. Schwinger, Physical Review Letters 18, 923 (1967); J. Wess and B. Zumino, Physical Review 163, 1727 (1967).

4. a. S. Adler, Physical Review 140, B736 (1965);
   b. W. Weisberger, Physical Review 143, 1302 (1966).

5. S. Fubini, Nuovo Cimento 43A, 475 (1966).
   In the non-local limit for which $q \to 0$, the procedure of the text involves no different assumptions than those of Ref. 4.

6. Incidentally, if one had another, independent determination of $G_A^2 = 1$ one could turn this procedure around to determine the proton matrix element of two conserved axial charges commuted at equal times: $G_A^2 = 1 = \langle p | [A^+, A^-] | p \rangle$.

7. M. Gell-Mann, Physical Review 125, 1067 (1962); Physics 1, 63 (1964).

8. S. L. Adler, Physical Review 137, B1022 (1965).

9. a. S. Weinberg, Physical Review Letters 17, 616 (1966).
   b. For the published example see,
   N. Khuri, Physical Review 153, 1477 (1967).

10. ibid; Equations (16) and (20) determine the coefficient of $\delta_{ab}\delta_{cd}$. In the text, we use the boson state normalization $\langle p | p \rangle = (2\pi)^3\, 2p_0\, \delta(\underset{\sim}{0})$.

11. The boson-boson invariant amplitude is defined from the S-matrix by:

$$S = 1 - \frac{i(2\pi)^4 \delta(p' + q' - p - q)}{(2\pi)^6 \sqrt{16 p_0' p_0 q_0' q_0}} A$$

12. The literature dealing with unitarity corrections to the $\pi\pi$ scattering length determination of Weinberg is rather lengthy. A representative article, L. S. Brown and R. L. Goble, Physical Review Letters 20, 346 (1968), constructs a unitarized form of the $\pi$-$\pi$ amplitude which is constrained at threshold by the current algebra result. The low-energy result can then be extended to a higher range of energies: the s-wave I = 0 and I = 2 phase shifts vary approximately linearly up to $\sqrt{s} \approx m_K$ where $\delta^{I=0} \approx 20^\circ$ and $\delta^{I=2} \approx -12^\circ$. A recent extrapolation procedure was made by W. D. Walker et. al., Physical Review Letters 18, 630 (1967), in which an increasing $\delta^{I=0}$ rises from $\approx 35^\circ$ at $\sqrt{s} \approx m_K$ and goes through $90^\circ$ in the neighborhood of $\sqrt{s}$ = 850 to 950 MEV so that a broad I = 0, s-wave resonance is indicated. Meanwhile $\delta^{I=2}$ smoothly decreases with energy to $\approx -20^\circ$ in the region $\sqrt{s}$ = 625 to 875 MEV.

13. From eqn. (22) which relates A to the current amplitude F and the current identity (20), one can see that taking $q \to 0$ (while keeping the other particles on mass-shell) in the subsequent general, current algebraic expression for A takes out both equal-time commutators due to a $(q^{2\prime} - m_\pi^2)$ factor.

14. That is, even though there are large mass differences in the pseudo-scalar octet, the <u>states</u> could be almost symmetric,

thus allowing construction of the (successful) octet-broken mass formula in the first place.

15. In a mathematical limit for which $\varkappa = -\sqrt{2}$, one sees from eqn. (13) that there results the invariant SU(2) $\times$ SU(2) algebra of isotopic vector and axial-vector charges (in the prototype quark model such a limit corresponds to a vanishing iso-doublet mass, $m_d = 0$). If $\varkappa = 2\sqrt{2}$ then there is a mixed parity SU(3) invariance, while the mathematical limit $\varkappa = 0$ gives the familiar SU(3) algebra of conserved vector charges, etc.

16. In the decay rate ratio,

$$\Gamma(K^+\to\mu\nu)/\Gamma(\pi^+\to\mu\nu) = \left(\frac{f_K}{f_\pi}\right)^2 \tan^2\theta_A \left(\frac{m_K}{m_\pi}\right)\left[\left(1 - \frac{m_\mu^2}{m_K^2}\right)/\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)\right]^2 ,$$

we use the assumption of a "universal" Cabibbo angle $\tan\theta_A = \tan\theta_\beta$. $\theta_\beta$ includes radiative corrections to $G_V = G_\mu \cos\theta_\beta$, which is obtained from $\beta$-decay assuming a point-like Fermi interaction. We use $\tan\theta_\beta = .209$ (C. S. Wu, Review of Modern Physics 36, 618 (1964)), while $\Gamma(K^+ \to \mu\nu)/\Gamma(\pi^+ \to \mu\nu) \simeq 1.31$ (A. H. Rosenfeld, et. al., Review of Modern Physics 37, 633 (1965)). Then it is found that $|f_K/f_\pi| \simeq 1.30$. In Cabibbo theory where the vector currents are unrenormalized in the SU(3) symmetric limit, it is natural to take $|f_K/f_\pi| \cong 1$.

Cabibbo (Proceedings of 13th International Conference on High Energy Physics, p. 33, University of California, 1967.) has also used the assumption $\theta_A = \theta_\beta$ and he points out that fits from hyperon decays determine the quantity $\frac{f_K}{f_\pi} \cdot \tan\theta_\beta \simeq .274$. Again, $\theta_\beta$ includes radiative corrections and may be checked with the value obtained from $K_{e_3}$ decay ($\sin\theta_\beta \simeq 0.21$) so that $f_K/f_\pi \simeq 1.28$. It should also be mentioned that the theoretical determination $f_K/f_\pi = 1.16$ results from single-particle saturation of vector and axial-vector spectral functions (S. L. Glashow, H. J. Schnitzer and S. Weinberg, Phys. Rev. Letters 19, 139 (1967).)

17. It should perhaps be pointed out that if one were to take limits $p', q' \to 0$ while leaving the initial pion and kaon on mass shell (without taking the necessary additional energy conserving limit $p^2 \to m_\pi^2$ or, alternatively, $q^2 \to m_K^2$), two additional equations would be obtained whose right-hand sides are zero due to the supposed absence of $I = 3/2$ components (SU(3) 27

transformation contributions) in the relevant commutators. These equations turn out to be consistent with solutions (31) by substitution. Taking three particles off mass shell for energy conservation, one again has to evaluate scalar density terms. The variation from $p^2 = m_K^2$ to $m_\pi^2$ involves the momentum transfer to one of these scalar terms, which, if it is slowly varying, explains the former naive limit where energy conservation was ignored.

18. In terms of the classic amplitude $f = \sum_\ell (2\ell+1) P_\ell(\cos\theta) \frac{e^{i\delta_\ell} \sin\delta_\ell}{q_{CM}}$ which satisfies $\frac{d\sigma^{\text{elastic}}}{d\Omega} = |f|^2$ and $\text{Im} f(\sqrt{s}, 0) = \frac{q_{CM}}{4\pi} \sigma_T$,

$$A = -8\pi\sqrt{s}\, f \, .$$

Hence at threshold where $q_{CM} = 0$ and $\sqrt{s} = (M+m)$ for elastic scattering of particles of masses M and m,

$$A^I_{th} = -8\pi(M + m)a^I \, .$$

For identical particles with an additional (statistical) factor of two as in the KK example within,

$$A^I_{th} = -32\pi\, m_K\, a^I \, .$$

19. We sketch the experimental situation. G. Goldhaber, 13th Conference on High Energy Physics p. 123 (1966), reports a definite $K_1^o K_1^o$ enhancement at or around threshold as studied in $\pi^- p \to K_1^o K_1^o n$. The $I = 0$, $J^{PG} = 0^{++}$ state(s) can be one or both of the following i) Resonance; $E(K_1K_1) = 1068$ MEV, $\Gamma(K_1K_1) = 80$ MEV, ii) Complex scattering length; $a = 2 + 0.12i$ to $3.3 + 0.2i$ F(fermis); G. Goldhaber, Proceedings of Second Hawaii Topical Conference in Particle Physics (1967), reports in addition a

threshold $K\bar{K}$ I = 1 state in $p\bar{p} \to K_1{}^o K^{\pm} \pi^{\mp}$. Fits which are all about equally good are: i) Positive real scattering length, a = 2.5 ± 1 F; ii) Complex scattering length, a = $-2.3 + i\,(0^{+.5}_{-.0})$ F; and iii) Resonance just above threshold, E = 1016 ± 10 MEV.

20. a. C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 193 (1966); V. S. Mathur, S. Okubo and L. K. Pandit, Phys. Rev. Letters 16, 371 (1966). The form factor $f_+(\Delta^2)$ in equation (46) of the text is normalized so that if there were exact SU(3) symmetry, $f_+(0) = 1$ (and $f_- = 0$). There are a variety of ways for satisfying (46) and, so far, many values of $\xi = f_-(0)/f_+(0)$ are possible, including negative ones. See in particular, N. Cabibbo, Proceedings of 13th International Conference on High Energy Physics (University of California Press, 1967) p. 33.

b. Another way to derive the Callan-Treiman relation is to start with the matrix element $\langle \pi^o(p) | \partial^\mu V_\mu^{K^-} | K^+(k) \rangle$ and use scalar and pseudo-scalar density commutation relations as well as the equation for $\varkappa$:

$$\langle \pi^o(p) | \partial^\mu V_\mu^{K^-} | K^+(k) \rangle = \frac{i}{2} \left[ (m_K{}^2 - p^2) f_+ + \Delta^2 f_- \right],$$

but also,

$$\langle \pi^o(p) | \partial^\mu V_\mu^{K^-} | K^+(k) \rangle = -i \frac{\sqrt{3}}{2} \varkappa \langle \pi^o | u^{K^-} | K^+ \rangle$$

$$= \left(\frac{3}{2}\right)^{1/2} \varkappa \cdot \frac{(m_\pi{}^2 - p^2)}{m_\pi{}^2 f_\pi} \int dx^4\, e^{ip\cdot x} \langle 0 | T\left\{ D^{\pi^o}(x) u^{K^-}(0) \right\} | K^+(k) \rangle$$

$$\underset{(p \to 0)}{\to} -\left(\frac{3}{2}\right)^{1/2} \frac{\varkappa}{f_\pi} \langle 0 | [A^{\pi^0}, u^{K^-}] | K^+ \rangle = \frac{i}{2}\left(\frac{3}{2}\right)^{1/2} \frac{\varkappa}{f_\pi} \langle 0 | v^{K^-} | K^+ \rangle$$

$$= -\frac{3i}{4}\left(\frac{\varkappa}{\sqrt{2} - \frac{\varkappa}{2}}\right) \frac{f_K}{f_\pi} m_K^2 .$$

Referring again to the vector current divergence matrix element in the limit $p \to 0$ ($\Delta^2 \to m_K^2$),

$$3\left(\frac{-\varkappa/\sqrt{2}}{2-\varkappa/\sqrt{2}}\right)\left(\frac{f_K}{f_\pi}\right) = [f_+(m_K^2, 0, m_K^2) + f_-(m_K^2, 0, m_K^2)] .$$

With $\varkappa \approx -\sqrt{2}$, the LHS of this equation is just $f_K/f_\pi$.

21. S. Weinberg, Phys. Rev. Letters <u>17</u>, 336 (1966); Phys. Rev. Letters <u>18</u>, 1178 (1967).

22. S. M. Berman and P. Roy, Phys. Letters <u>27B</u>, 88 (1968).

23. ibid; If $\xi = f_-(m_K^2, m_\pi^2, \Delta^2)/f_+(m_K^2, m_\pi^2, \Delta^2) \approx -1$, large variations in (especially) the pion mass-squared variable $p^2$ would have to be assumed. For the $K_{L_3}$ vertex, the origin of such possible variation is unclear and, if present, would make implausible the smoothness property (used in this reference) for $F_1$ and $F_2$ as well. Section IV does not <u>require</u> $\xi \approx -1$.

24. J. Iliopoulos and R. Van Royen, Phys. Letters <u>25B</u>, 146 (1967).

25. In order to estimate $|F_3(m_K^2, m_\pi^2, 0, k \cdot p, 0, 0)| \leq 0.15$ (small), the authors of Ref. (22) use a cutoff and attempt to saturate an axial current, divergence commutator matrix element to which $F_3$, in the indicated limit, is proportional:

$$F_3(m_K^2, m_\pi^2, 0, k\cdot p, 0, 0)$$

$$= -i\sqrt{2}\, m_K f_K f_\pi^{-1} \int dx^4 \langle \pi^+(p) | [A_0^{\pi^+}(x), \phi^{K^-}(0)]\, \delta(x_0) | K^+(k)\rangle .$$

We may directly take this expression and, employing the techniques of Section III, use the behavior $(3, 3^*) + (3^*, 3)$ of scalar and pseudo-scalar densities $u^i, v^i$ under $SU(3) \times SU(3)$ to evaluate the commutator. Then the low-energy limit for the pion $\pi^+(p)$ is applied to the subsequent scalar vertex (where it is assumed that the $p \to 0$ limit does not appreciably change such a matrix element):

$$F_3(m_K^2, m_\pi^2, 0, k\cdot p, 0, 0)$$

$$= -i\sqrt{2}\, m_K f_K f_\pi^{-1} \int dx^4 \langle \pi^+(p) | [A_0^{\pi^+}(x), g\, v^{K^-}] | K^+\rangle$$

$$= m_K f_K f_\pi^{-1} g \langle \pi^+(p) | u^{\bar{K}^0} | K^+\rangle$$

$$= -i\sqrt{2}\, m_K f_K f_\pi^{-2} \langle 0 | [A^{\pi^+}, u^{\bar{K}^0}] | K^+\rangle$$

$$= -m_K f_K f_\pi^{-2} \langle 0 | g\, v^{K^-} | K^+\rangle = -m_K f_K f_\pi^{-2} . \qquad (60)'$$

This result is to be contrasted with $F_3(m_K^2, m_\pi^2, 0, k\cdot p, 0, 0) \approx 0.$[22)]

26. The linear combination $F_{45}^{(+)} + F_{67}^{(+)}$ was taken so as to eliminate the matrix element $\langle p | u_3 | p\rangle$ not present in a matrix element of (1): $F_{45}^{(+)}(0,0,0,0) = \frac{2}{3}(\sqrt{2} - \frac{\varkappa}{2})\langle p | \sqrt{2}\, u_0 - \frac{1}{2} u_8 + \frac{\sqrt{3}}{2} u_3 | p\rangle$ and $F_{67}^{(+)}(0,0,0,0) = \frac{2}{3}(\sqrt{2} - \frac{\varkappa}{2})\langle p | \sqrt{2}\, u_0 - \frac{1}{2} u_8 - \frac{\sqrt{3}}{2} u_3 | p\rangle$ .

27. R. W. Griffith, Calt-68-136 (unpublished Caltech report).

28. The Born piece $T_{p\pi}^{(+)B}$ has been calculated in the usual manner from the one-particle contributions in the invariant amplitudes A and B: $T(\nu, t, m_\pi^2, 0) = \bar{u}(p')\{-A+(\gamma \cdot K)B\}u(p)$. The result is the perturbation answer.

29. For the boson-fermion amplitude the invariant amplitude is defined by

$$S = 1 - \frac{i(2\pi)^4 \delta(p' + q' - p - q)}{(2\pi)^6 \sqrt{4 \frac{p_0'}{M} q_0' \frac{p_0}{M} q_0}} \cdot T$$

30. C. Lovelace, Proceedings of the Heidelberg International Conference on Elementary Particles, p. 81, 1968.

31. K. Kawarabayashi and W. W. Wada, Phys. Rev. 146, 1209 (1966).

32. Indeed, suppose one were to think of the physical situation as being approximately $SU(2) \times SU(2)$ invariant by virtue of a small pion mass and $\kappa$ close to $-\sqrt{2}$. One could not, by the same argument, consider the kaon divergence $D^K$ to be "small" because the relevant factor there is $(\sqrt{2} - \frac{\kappa}{2})$. In fact, eqn. (3) demonstrates how large is the ratio of proportionally factors in the divergences $D^\pi$ and $D^K$. In terms of the geometrical statement of pole dominance, the next threshold after the pion in the pion channel is at $(3m_\pi)^2$ while the analogous situation for the kaon channel occurs at $(m_K + 2m_\pi)^2$.

33. R. Phillips and W. Rarita, Phys. Rev. 139, B1341 (1965). Solution I of this reference corresponds to $C_P/\alpha_P = 2(17.70)$, $C_{P'}/\alpha_{P'} = 2(10.94)$ mb $\times$ BEV and $\alpha_P = 1$, $\alpha_{P'} = .5$ with scale factor $\nu_0 = 1$ BEV. Equation (74) is not sensitive to the choice of solutions.

34. a. J. K. Kim, Phys. Rev. Letters 14, 29 (1965);
b. S-wave scattering lengths obtained from an effective range analysis in the following references only slightly change the evaluation of eqn. (74).
J. K. Kim, Phys. Rev. Letters 19, 1074 (1967);
F. Von Hippel and J. K. Kim, Phys. Rev. Letters 20, 1303 (1968).

35. R. Dashen, Y. Dothan, S. Frautschi and D. Sharp, Phys. Rev. 143, 1185 (1966). When by SU(3) invariance $G^2_{Y_1^{o*} pK^-} = \frac{1}{6} G^2_{N_*^{++} p\pi^+}$, this reference estimates in broken SU(3) $G^2_{Y_1^{o*} pK^-} = .117\, G^2_{N_*^{++} p\pi^+}$ (from $\Gamma = \frac{G^2}{12\pi} \frac{M}{M_*} p^3_{CM} (\frac{E}{M} + 1) = 120$ MEV, $G^2_{N_*^{++} p\pi^+} = 236\ (\mathrm{BEV})^{-2}$).
Actually, more accurate multi-channel analyses will probably indicate more suppression of the $Y_1^* \bar{K}N$ channel. When by SU(3) invariance $g^2_{K\Lambda N} = \frac{1}{3}(3 - 2\alpha)^2 g^2_{\pi NN}$ and $g^2_{KN\Sigma} = (1 - 2\alpha)^2 g_{\pi NN}$ with $\alpha \simeq \frac{2}{3}$, the broken estimates are $g^2_{K\Lambda N} = .395\, g^2_{\pi NN}$ and $g^2_{K\Sigma N} = .0775\, g^2_{\pi NN}$.

36. J. K. Kim, Phys. Rev. Letters 19, 1079 (1967).

37. Adler's consistency condition on $\pi N$ amplitudes (Phys. Rev. 137, B 1022 (1965)) cannot provide unambiguous information about scalar terms because it applies when the four-momenta of one pion are zero and the other particles are all on-mass-shell:

$$\bar{u}(p')[A + \gamma \cdot \frac{1}{2}(q + q')B]u(p) = \frac{(m_\pi^2 - q^{2\prime})}{m_\pi^2 \frac{f_\pi}{\sqrt{2}}} q^{\mu\prime} \langle N|A_\mu^\pi|N\pi\rangle \quad \text{for small } q'$$

so

$$\bar{u}(p')[A + \gamma \cdot \frac{1}{2}(p' - p)B]u(p) = \bar{u}(p')u(p) \cdot A = 0 \quad \text{at } q' = 0$$

and $$A^{(+)}(\nu=0,\ t=m_\pi^2,\ w=\frac{m_\pi^2}{2},\ v=\frac{m_\pi^2}{2}) = 0 .$$

In connecting the strong $\pi N$ amplitude $A^{(+)}$ with the weak amplitude $F^{(+)}$ in the current identity (62), the factor $(q^2 - m_\pi^2)(q^{2\prime} - m_\pi^2)$ is involved which, in the limit $q' \to 0$, $q^2 = m_\pi^2$, eliminates the scalar density commutator from consideration. "Smoothness" arguments for extrapolation in the pseudo-scalar mass variable lack a quantitative basis (at the very least) due to complications in the pion case of the factor $(\sqrt{2} + \kappa)$ connecting the scalar terms and the commutator actually appearing in (62). In the kaon case, of course, one must know just how large is the extrapolation from $q^2 = m_K^2$ to $q^2 = 0$.

38. It will be understood that vacuum expectation values are subtracted out so that, for example, we really deal with

$$f^{ij} = -\frac{1}{2}\int dx^4\, e^{iq\cdot x}\left\{ \langle p|[D^i(x), D^j(0)]|p\rangle - \langle 0|[D^i(x), D^j(0)]|0\rangle \right\}.$$

This procedure eliminates an uninteresting, totally disconnected set of graphs from the scattering amplitudes.

39. From W. Weisberger, Phys. Rev. Letters 14, 1047 (1965), as well as just the form of (D-2), this procedure is equivalent to writing the connected part as a sum over source current matrix elements for the $J^P = 0^-$ resonances, divided by propagators and multiplied by the appropriate normalizations $\langle 0|D|\beta\rangle$.

Of course, the mass dispersion relation applied to the unitary spin anti-symmetric case gives the Adler-Weisberger sum rule.

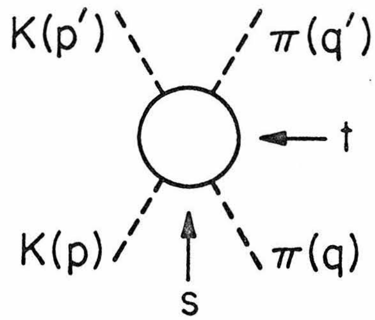
40. $\tilde{T}$ denotes the non-Born amplitude and, although for general meson scattering it may be ambiguous, here for $J^P = 0^-$ resonances it is defined just as in Ref. (28).

41. S. Fubini and G. Furlan, "Dispersion Theory of Low Energy Limits," Annals of Physics 48, 322 (1968).

D. Amati, R. Jengo and E. Remiddi, Nuovo Cimento 51A, 999 (1967), first tried to exploit a representation like (81) (at infinite momentum, $p_0 \to \infty$). High energy ($\nu$) and high mass ($q^2$) assumptions for the hadron production graphs have to be made when connecting this kind of representation to the scalar density terms (R. W. Griffith, unpublished). Actually, similar problems arise in the Breit frame, finite $p_0$ application of the first authors above, because the $q_0 \to \infty$ limit ($\nu = q_0 \to \infty$, $q^2 = q_0^2 \to \infty$) produces an arbitrary subtraction constant in the final representation (84) for $F(q_0 = 0) =$ scalar density term.

42. V. J. Stenger et al., UCLA Report 1002;
S. Goldhaber et al., Phys. Rev. Letters 9, 135 (1962).

43. In fact, if one saturates the integral in (85) only by the $Y_0^*$ (1405, $1/2^-$) using a broken SU(3) value for the $Y_0^*$ $K^-p$ coupling (C. Weil, Phys. Rev. 161, 1682 (1967); $r = g^2_{\bar{K}N}/g^2_{\Sigma\pi} = 6.9$), then the integral contributes (-1500) MEV! Of course, this saturation is very sensitive to the threshold factor $(\nu_*^2 - m_K^2)^{-1} \simeq (\frac{1}{5} m_K^2)^{-1}$.




$$s = (p+q)^2$$

$$t = (p' - p)^2$$

$$u = (p - q')^2$$

$$\text{masses} = q^2, q^{2'}, p^2, p^{2'}$$

Fig. 1

Off-mass-shell pseudo-scalar scattering amplitude

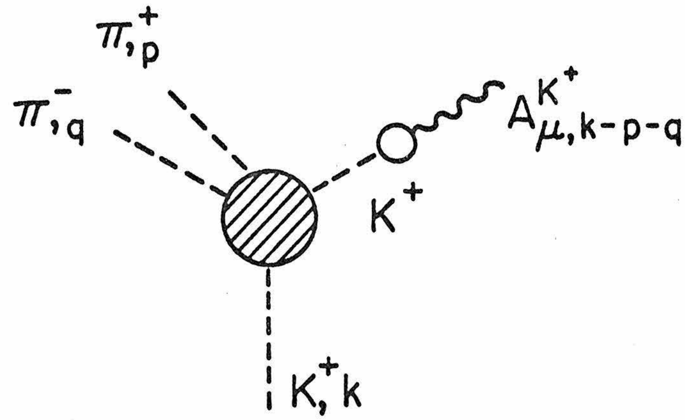
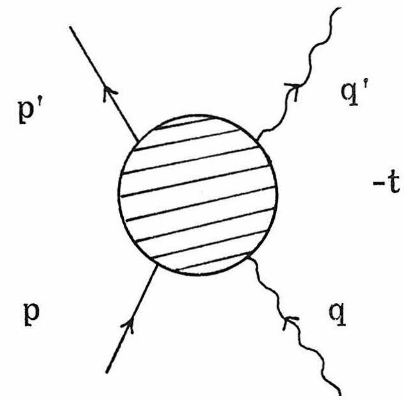



Figure 2. $K_{L_4}$ kaon-pole diagram.




$$P = \frac{1}{2}(p' + p)$$

$$K = \frac{1}{2}(q' + q)$$

$$\Delta = (p' - p) = (q - q')$$

masses: $p^2 = p'^2 = M^2,\ q^2,\ q'^2$

invariants: $\nu = \dfrac{P \cdot K}{M},\ t = \Delta^2,\ w = \frac{1}{2}(q^2 + q'^2),\ v = \frac{1}{2}(q^2 - q'^2)$

Figure 3. Kinematical Definitions

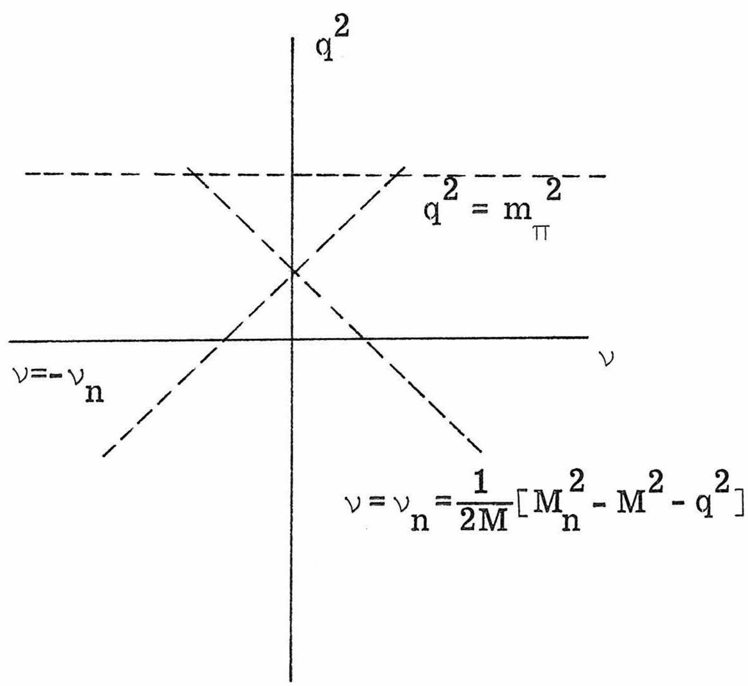



Figure 4-a. Some Singularities of Weak Amplitude $F(\nu, q^2)$.

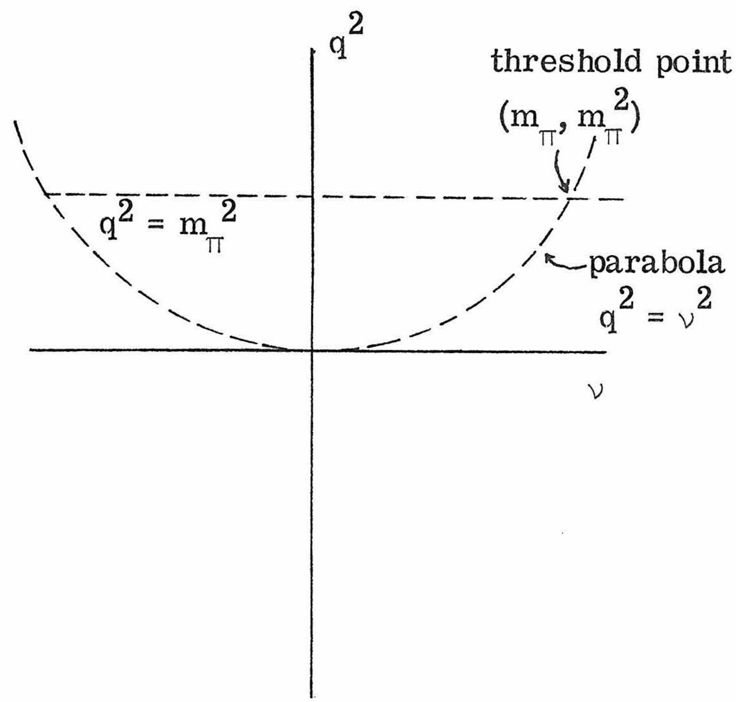
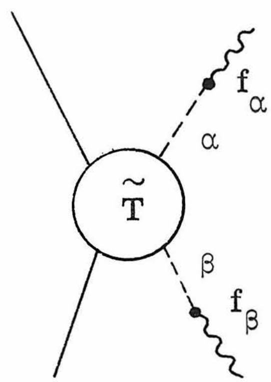



Figure 4-b. Fubini-Furlan Breit Frame Parabola for $F(q_0)$.

$$\frac{1}{\pi}\int \frac{dq^2}{q^2}\,\tilde{f}(q^2,0) = \sum_{\alpha,\beta}$$




$$= \sum_{\alpha,\beta} \frac{(m_\alpha^2 f_\alpha)(m_\beta^2 f_\beta)\,\tilde{T}(p\alpha;\, p\beta)}{(m_\alpha^2 - q^2)(m_\beta^2 - q^2)}\,, \quad q^2 = 0\,.$$

Figure 5. Mass Dispersion Relation and Amplitude Residues